A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

1. Consider $L^2(\mathbb{Z}) := H$. Recall two topologies on the set of bounded linear operators on $H$: The norm topology which is metric topology given by $d(A, B) = \sup_{v \in H: ||v|| = 1} ||Av - Bv||$. The strong operator topology which is given by $A_i \rightarrow A_\infty$ if for every $v \in H$ we have $\lim_{i \rightarrow \infty} ||A_i v - A_\infty v|| = 0$.

   (a) Show that if $A_i$ converges to $A_\infty$ in the norm topology then it converges to $A_\infty$ in the strong operator topology.
   (b) Give an example of $A_i$ which converges to $A_\infty$ in the strong operator topology but not the norm topology.
   (c) Let $A_i$ be the operator defined by $A_i v = w$ where $w_j = v_j$ if $j \neq i$ and $w_i = iv_i$. Show $A_1, ...$ is a sequence of operators that does not converge in either topology.

2. Let $(X, \mathcal{B}, \mu)$ be a measure space. $U \in \mathcal{B}$ is called an atom if $\mu(U) > 0$ and $\mu(V) = 0$ for all $V$, a proper subset of $U$. If $(X, \mathcal{B}, \mu)$ is $\sigma$-finite measure space and $f : X \rightarrow Y$ then $f_*(\mu)$ has at most countably many atoms.

3. Let $f_i : [0, 1] \rightarrow \mathbb{R}$ be a sequence of functions that converge pointwise to $f_\infty$.
   (a) Show that if the $f_i$ are uniformly bounded then $\lim_{i \rightarrow \infty} \int f_i d\lambda = \int f_\infty d\lambda$.
   (b) What if $\|f_i\|_1 \leq 1$ for all $i$?
   (c) What if $\|f_i\|_2 \leq 1$ for all $i$?

4. Let $f \in L^1(\lambda, [0, 1])$ have the property that for any measurable set $A$ with $\lambda(A) = \frac{1}{\pi}$ we have $\int_A f d\lambda = 0$. Show that $f = 0$ almost everywhere.

5. Let $A \subset \mathbb{R}$ and $A + A$ denote $\{a_1 + a_2 : a_1, a_2 \in A\}$. Show that if $\lambda(A^c) = 0$ then $A + A = \mathbb{R}$.
B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

6. Let \( f \) be a nonzero entire function. Assume that the set \( Z(f) = \{ z \in \mathbb{C} : f(z) = 0 \} \) is unbounded. Show that \( f \) had an essential singularity at \( \infty \).

7. Let \( f \) be an holomorphic function defined on an open set \( U \subset \mathbb{C} \) and \( a \in U \). Assume that \( f'(a) \) is nonzero. Show that for sufficiently small circles \( C \) centered at \( a \)

\[
\frac{2\pi i}{f'(a)} = \int_C \frac{dz}{f(z) - f(a)}
\]

8. Let \( f \) be an entire function and \( n \) a positive integer. Show that there exists an entire function \( g \) such that \( f = g^n \) if and only if the order of each zero of \( f \) is divisible by \( n \).

9. Show that all the zeroes of \( f(z) = z^4 + 6z + 3 \) lie inside the circle \( |z| = 2 \).

10. Evaluate

\[
\int_C \frac{1}{(z + i)(z - 1)} \, dz
\]

where \( C \) is pictured on the next page: