Instructions: Do seven problems and list on the front of your blue book the seven problems to be graded. Do at least three problems from each part.

Part A:

Problem 1. Suppose \((X, M, \mu)\) is a measure space and \(f : X \to \mathbb{R}\) is a real-valued function on \(X\). Suppose further that \(E_r := \{x \mid f(x) > r\}\) is measureable for each rational number \(r\). Either prove the following assertion or find a counter-example: \(f\) is measureable.

Problem 2. Suppose \((X, M, \mu)\) is a measure space and fix \(p\) and \(q\) finite such that \(\frac{1}{p} + \frac{1}{q} = 1\). Let \(f_1, f_2, \ldots\) be a sequence of functions in \(L^p(X)\) converging (in \(L^p\)) to \(f\), and let \(g_1, g_2, \ldots\) be a sequence of functions in \(L^q\) converging (in \(L^q\)) to \(g\). Prove that the sequence \(f_1 g_1, f_2 g_2, \ldots\) converges to \(fg\) in \(L^1\). Does the same conclusion hold if \(p = 1\) and \(q = \infty\)?

Problem 3. Let \(H\) be a Hilbert space and suppose that \(\{x_n\}\) is a sequence in \(H\) with the following property: for each \(y \in H\),

\[
\sup_n |\langle x_n, y \rangle| < \infty.
\]

Prove that \(\sup_n ||x_n|| < \infty\).

Problem 4. Suppose \(1 < p < q < r < \infty\). (Here \(p\) and \(q\) are arbitrary, not necessarily conjugate.) Prove that \(L^p(\mathbb{R}) \cap L^r(\mathbb{R}) \subset L^q(\mathbb{R})\).

Problem 5. Let \(H\) be a Hilbert space, \(M\) a closed subspace of \(H\), and \(x \in H\). Prove that there is a unique point \(y \in M\) which is closest to \(x\).
Part B:

Problem 6. Let \( f(z) = 1 - \cos z. \)

(i) Find all zeros of this function;
(ii) Find the multiplicities of these zeros.

Problem 7. Let \( f(z) = \sin \left( \frac{z}{z + 1} \right). \)

(i) Determine all isolated singularities of \( f \) and their type;
(ii) Find the Laurent expansions of \( f \) at these singularities;
(iii) Find the residues of \( f \) at these singularities.

Problem 8. Evaluate the integral
\[
\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 - 2x + 10} \, dx
\]
using the residue theorem.

Problem 9. Let \( n \) be a positive integer. Denote by \( V_n \) the linear space of all entire functions \( f \) such that there exists \( C > 0 \) such that \( |f(z)| \leq C|z|^n \) for all \( z \in \mathbb{C} \).

(i) Describe precisely the functions in \( V_n \);
(ii) Find the dimension of \( V_n \).

Problem 10. Using Rouche’s theorem find the number of zeros of the polynomial \( 2z^5 - z^3 + 3z^2 - z + 8 \) in the region \( \{ z \mid |z| > 1 \} \).