DEPARTMENT OF MATHEMATICS

University of Utah
Ph.D. PRELIMINARY EXAMINATION IN ANALYSIS
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Instructions: Do seven problems with at least three (3) problems from section A and three (3) problems from section B. You need at least two problems completely correct from each section to pass. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems answered will be scored.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

1. Let $\{f_n\}$, f be measurable functions on a measure space (X, μ) such that

$$f_1 \geq f_2 \geq \cdots \geq f \geq 0$$

Suppose that $\int f_n \, d\mu \to \int f \, d\mu$ and that f_1 is integrable. Prove that $f_n \to f$ almost everywhere. Show by example, that this conclusion may be false if f_1 is not integrable.

2. Recall that the Fourier transform of a function $f \in L^1(\mathbf{R})$ (with respect to Lebesgue measure) is given by

$$\hat{f}(x) = \int_{-\infty}^{\infty} e^{-2\pi i x y} f(y) \, dy$$

Show that $\hat{f}$ is a bounded, continuous function on $\mathbf{R}$.

3. Let H be a Hilbert Space, and suppose that $\{x_n\}$ is a sequence in H such that $< x_n, y >$ is convergent for each $y \in H$. Prove that there exists $x \in H$ such that $< x_n, y > \to < x, y >$ for all $y \in H$.

4. Let (X, μ) be a finite measure space, and let $f \in L^\infty(X)$. For $g \in L^1(X)$, define

$$Tg = \int f g \, d\mu$$

Show that T is a bounded linear functional on $L^1(X)$, and that $\|T\| = \|f\|_\infty$.

5. Either prove the following statement if true, or prove that it is false by providing a counterexample. Let $\{f_n\}$ be a sequence in $L^2([0, 1])$ such that $\|f_n\|_2 \to 0$. Then $f_n \to 0$ almost everywhere.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

6. Prove Schwarz's lemma: If $f : \Delta \to \Delta$ is a holomorphic function from the open unit disk Δ to itself with $f(0) = 0$ show that $|f(z)| \leq |z|$, $|f'(0)| \leq 1$ and if $|f(z)| = |z|$ for some $z \neq 0$ or $|f'(0)| = 1$ then $f(z) = \lambda z$ with $|\lambda| = 1$.

   (Hint: Apply the maximum principle to the function $f(z)/z$.)

7. Calculate
$$\int_\gamma (1 + z^2) e^{1/z} dz$$
   where γ is the unit circle traversed in the counter-clockwise direction.

8. Let $f : \mathbb{C} \to \mathbb{C}$ be a function such that:

   (a) f is continuous;

   (b) f is a holomorphic at all points z not on the real axis;

   (c) if z is on the real axis then $f(z)$ is real.

   Show that $f(\bar{z}) = \overline{f(z)}$ and that f is holomorphic on all of $\mathbb{C}$.

9. Let f be a function holomorphic for z in the annulus $r < z < R$. Show that if for some ρ with $r < \rho < R$,
$$\int_{|z|=\rho} z^n f(z) dz = 0$$
   for all integers n with $n \geq 0$ then f extends to a holomorphic function on the disk $|z| < R$.

10. Show that all zeroes of $z^4 - 6z - 3$ lie inside the circle $|z| = 2$.