Departement of Mathematics
University of Utah
Real and Complex Analysis Qualifying Exam

Instructions: Do seven problems, at least three from part A and three from part B. List the problems you have done on the front of your blue book.

Part A.

1. Let $\ell^1$ be the Banach space of real sequences $a = (a_1, a_2, \ldots)$ with norm $\|a\|_1 = \sum_{i=1}^{\infty} |a_i| < \infty$, and $\ell^\infty$ the Banach space of real sequences with norm $\|x\|_\infty = \sup |x_i|$. Let $a \in \ell^1$ be a fixed sequence; for any sequence $x \in \ell^\infty$, define a new sequence $T_a(x)$ by

$$(T_a(x))_n = \sum_{i=1}^{n} a_i x_i$$

(a) Show that this defines a bounded operator $T_a : \ell^\infty \to \ell^\infty$.

(b) Show $\|T_a\| = \|a\|_1$, where the left-hand side denotes the norm of the operator $T_a : \ell^\infty \to \ell^\infty$.

2. Let $(\mathcal{M}, X, \mu)$ be a finite positive measure space.

(a) Show that $L^2(X) \subset L^1(X)$, and that the inclusion $i : L^2(X) \subset L^1(X)$ has norm $\|i\| = \sqrt{\mu(X)}$.

(b) Let $\mathcal{M}' \subset \mathcal{M}$ be the subset of measurable sets $E$ with $\mu(E) > 0$. Show that if

$$L^1(X) = L^2(X),$$

then

$$\inf_{E \in \mathcal{M}'} \mu(E) > 0.$$  

3. Let $H$ be a Hilbert space with inner product written as $\langle f, g \rangle$, and let $f, g \in H$ be two non-zero elements. Show that $f = zg$ for some $z \in \mathbb{C}^*$ if and only there exists no $h \in H$ with

$$\langle f, h \rangle = 1 \quad \text{and} \quad \langle g, h \rangle = 0.$$  

4. Let $f : [0, 1] \to \mathbb{R}_{\geq 0}$ be a measurable integrable function $f \in L^1([0, 1])$ (with respect to the Lebesgue measure). Show that

$$\int_{[0, 1]} xf(x)dx = \int_{[0, 1]} \left( \int_{[y, 1]} f(x)dx \right) dy$$

5. Let $T : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ be the translation operator defined by $(Tf)(x) = f(x+1)$. Show that there is no non-zero bounded linear functional $\Phi : C_0(\mathbb{R}) \to \mathbb{C}$ invariant under $T$, i.e. such that $\Phi$ satisfies

$$\Phi(f) = \Phi(Tf).$$
Part B. In the following, $D$ and $\overline{D}$ will denote the open and the closed unit disks, respectively.

6. Compute the integral
\[ \int_0^\infty \left( \frac{\sin x}{x} \right)^2 \, dx. \]

7. Let $a$ be an isolated singularity of the meromorphic function $f : \Omega \to \mathbb{C}$. Prove that if $a$ is an essential singularity, then in any neighborhood of $a$, the function $f$ takes values arbitrarily close to any complex number.

8. Let $\alpha > 1$ be arbitrary. Show that the equation
\[ \alpha - z - e^{-z} = 0 \]
has exactly one solution in the half plane $\{ z : \text{Re} \, z > 0 \}$, and moreover, this solution is real.

9. Let $\Omega \subset \subset \mathbb{C}$ be a simply connected region, and fix $z_0 \in \Omega$. If $\phi : \Omega \to D$ is a conformal map such that $\phi(z_0) = 0$, show that
\[ |\phi'(z_0)| = \sup\{|f'(z_0)| : f : \Omega \to D \text{ holomorphic}, f(z_0) = 0\}. \]

10. Let $f$ be a function holomorphic on $D$ and continuous on $\overline{D}$. Assume that $|f(z)| = 1$ whenever $|z| = 1$. Show that $f$ can be extended to a meromorphic function on the whole $\mathbb{C}$, with at most finitely many poles. 
   \begin{itemize}
   \item[(Hint:)] Starting with the Schwarz reflection principle for the upper half plane, deduce that an appropriate reflection continuation that can be used in this setting is $z \mapsto \frac{1}{f(1/z)}$.
   \end{itemize}