DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
REAL AND COMPLEX ANALYSIS EXAM

August 13, 2007

Instructions:. Do seven problems, at least three from part A and three from part B. List the problems you have done on the front of your blue book.

Part A.

1. Let f be integrable on $\mathbf{R}$ with respect to Lebesgue measure, and define

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Prove that F is continuous.

2. Assume that $\mathbf{R}$ has the Lebesgue measure. Let $f \in L^\infty(\mathbf{R})$. For $g \in L^2(\mathbf{R})$ define

$$M(g) = fg$$

Prove that $M : L^2(\mathbf{R}) \to L^2(\mathbf{R})$ is a bounded linear operator, and that $\|M\| = \|f\|_\infty$.

3. Let H be a complex Hilbert space, and let $T : H \to H$ be a bounded self adjoint operator (so $T = T^*$).

a. Prove that if λ is an eigenvalue of T, then λ is real.

b. Show that eigenvectors corresponding to different eigenvalues are orthogonal.

c. Assume in addition that T is compact. Show that if $\{\lambda_n\}$ is a sequence of distinct eigenvalues with $\lambda_n \to \lambda$, then $\lambda = 0$.

4. Let $f \in L^1([-\pi,\pi])$ and for $n \in \mathbf{Z}$ let c_n be the nth Fourier coefficient of f, that is

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

Prove that $c_n \to 0$ as $|n| \to \infty$.

5. Let $I = [0,1]$ be the unit interval with the Lebesgue measure, and let $I \times I$ be endowed with the product measure. Assume that $K \in L^2(I \times I)$ and $f \in L^2(I)$. Define

$$g(x) = \int_0^1 K(x,y) f(y)\,dy$$

Show that $g \in L^2(I)$.

Part B.

6. Let f be an entire function such that $|f(z)| \leq K|z|^n$ where K is a positive real constant and n is a positive integer. Show that f is a polynomial of degree $\leq n$.

7. Let $H = \{z \in \mathbf{C} | \Im z > 0\}$ be the upper half plane and let $\Delta = \{z \in \mathbf{C} | |z| < 1\}$ be the unit disk.

a. Find a conformal map from H to Δ that takes i to 0.

b. Let $f : H \to H$ be a holomorphic map with $f(i) = i$. Show that $|f'(i)| \leq 1$ and that if $|f'(i)| = 1$ then f is of the form $f(z) = \frac{az+b}{cz+d}$ with $a, b, c, d \in \mathbf{R}$.

8. Let f be an entire function such that for all $x \in \mathbf{R}$, $f(ix)$ and $f(1+ix)$ are in $\mathbf{R}$. Show that f is periodic with period 2. That is show that $f(z) = f(z+2)$ for all $z \in \mathbf{C}$.

9. Let Ω be an open connected subset of $\mathbf{C}$ and $\mathcal{F}$ a family of holomorphic functions with domain Ω such that every compact subset K of Ω there is a positive real constant M_K such that for all $f \in \mathcal{F}$ and $z \in K$ we have $|f(z)| < M_K$. Show that for every $z \in \Omega$ there exists a positive constant B_z such that $|f'(z)| < B_z$ for all $f \in \mathcal{F}$.

10. Calculate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 4} dx.$$