DEPARTMENT OF MATHEMATICS
University of Utah
Real and Complex Analysis Preliminary Examination
August 15, 2012

> **Instructions:** Do seven problems with at least three (3) problems from section A and three (3) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. List on the front of your blue book the seven problems to be graded.

A. Answer at least three and no more than four of the following questions. Each question is worth ten points.

1. Let V be a Banach space, and let $W \subset V$ be a 1-dimensional linear subspace. Prove that there exists a closed linear subspace $U \subset V$ such that $V = W \oplus U$.

2. Let (X, M) be a measurable space.

   (a) If μ is a positive measure on (X, M), $h \geq 0$ is a measurable function, and ν is the measure $\nu(A) = \int_A h \, d\mu$, show that $\int f \, d\nu = \int f h \, d\mu$, for all measurable functions $f \geq 0$.

   (b) If ν, μ, θ are positive measures on (X, M) with $\nu << \mu << \theta$, prove that $\frac{d\nu}{d\theta} = \frac{d\nu}{d\mu} \frac{d\mu}{d\theta}$. (Hint: use part a).

3. Let A and B be closed linear subspaces of a Hilbert space H, and assume that $A \perp B$. Prove that $A + B$ is closed in H.

4. Let (X, M, μ) be a measure space, and suppose that $f \in L^1(X, \mu)$. Prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that whenever $A \in M$ with $\mu(A) < \delta$, then

$$\left| \int_A f \, d\mu \right| < \epsilon$$

5. If the following statement is true, give a proof, if false, give a counterexample. If $f_n \in L^2([0,1], \mu)$ (with μ being the Lebesgue measure) and $\|f_n\|_2 \to 0$, then $f_n \to 0$ a.e.

B. Answer at least three and no more than four of the following questions so that the total number of questions you have answered is seven. Each question is worth ten points.

1. Find all zeros of the function

$$f(z) = \frac{\sin^3 z}{z}$$

and determine their orders. Explain your argument!

2. Find all isolated singularities of the function

$$f(z) = \frac{1 - e^z}{1 + e^z}$$

and determine their type (and the order in the case of poles). Find the residues of f at these isolated singularities. Explain your argument!

3. Using the residue theorem find the integral

$$\int_0^\infty \frac{x \sin(ax)}{(x^2 + b^2)^2} \, dx$$

where a and b are nonzero real numbers. Explain your argument!

4. Let c_1 and c_2 be two complex numbers which are linearly independent over real numbers. We say that a function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is double periodic with periods c_1 and c_2 if

$$f(z + nc_1 + mc_2) = f(z)$$

for any $z \in \mathbb{C}$ and $n, m \in \mathbb{Z}$.

Assume that such f is entire. What can you conclude about the function f?

5. Let f be a holomorphic automorphism of $\mathbb{C}^*$. Show that either $f(z) = \alpha z$ or $f(z) = \frac{\alpha}{z}$ for some complex number $\alpha \neq 0$.