Probability Qualifying Examination

January 7, 2011

There are 10 problems, of which you must turn in solutions for exactly 6 (your best 6, in your opinion). Each problem is worth 10 points, and 40 points is required for passing. On the outside of your exam book, indicate which 6 you have attempted.

If you think a problem is misstated, interpret it in such a way as to make it nontrivial.

1. Let $X_1, X_2, \ldots$ be independent identically distributed random variables with characteristic function $\varphi$. Let $N$ be a random variable with distribution

$$P\{N = k\} = \frac{1}{2^k}, \quad k = 1, 2, \ldots.$$ 

It is assumed that $\{X_i, i \geq 1\}$ and $N$ are independent.

(a) Compute the characteristic function of $Y = X_1 + \ldots X_N$.

(b) Can you weaken the condition that $\{X_i, i \geq 1\}$ and $N$ are independent so that the formula obtained in part (a) remains true?

2. (a) Let $\Phi$ and $\phi$ be the standard normal distribution and density functions, respectively. Show that

$$\lim_{x \to \infty} \frac{1 - \Phi(x)}{(1/x)\phi(x)} = 1.$$ 

(b) Let $X_1, X_2, \ldots$ be independent identically distributed standard normal random variables. Show that

$$\limsup_{n \to \infty} \frac{1}{\sqrt{\log n}} |X_n| = c \quad \text{almost surely}$$

and compute the value of $c$.

3. Let $X_1, \ldots, X_n$ be independent identically distributed random variables with $E X_1 = \mu$ and $0 < \text{Var}(X_1) = \sigma^2 < \infty$. Let

$$Y_n = \sum_{1 \leq i < j \leq n} X_i X_j.$$ 

Find numerical sequences $a_n$ and $b_n$ such that $(Y_n - a_n)/b_n$ has a non-degenerate limit distribution.
4. Let \(f\) be a continuous and bounded function on \([0, \infty)\). Compute

\[
\lim_{n \to \infty} \int_0^\infty \cdots \int_0^\infty f \left( \frac{x_1 + \cdots + x_n}{n} \right) e^{- (x_1 + \cdots + x_n)} \, dx_1 \cdots x_n.
\]

5. Let \(X_1, \ldots, X_n\) be independent identically distributed Poisson random variables for each \(n\) with parameter \(\lambda_n\).

(a) Show that \(X_1 + \cdots + X_n\) is asymptotically normal if and only if \(n\lambda_n \to \infty\).

(b) Can you weaken the condition that the \(X\)'s are identically distributed for each \(n\)?

6. (a) Let \(X\) be a random variable with characteristic function \(\varphi\). Show that \(X\) is symmetric if and only if \(\varphi(t)\) is real for all \(t\).

(b) Give three distinct examples of real characteristic functions.

7. Let \(X\) and \(Y\) be integrable random variables on \((\Omega, \mathcal{F}, P)\), and let \(\mathcal{G}\) be a sub-\(\sigma\)-algebra of \(\mathcal{F}\). If \(X = Y\) on \(G \in \mathcal{G}\), show that \(E[X \mid \mathcal{G}] = E[Y \mid \mathcal{G}]\) a.s. on \(\mathcal{G}\).

8. Let \(Z_n\) be a Galton–Watson branching process with offspring distribution \(\{p_k, \, k = 0, 1, 2, \ldots\}\) and \(Z_0 = x\) (with \(x\) a positive integer), and let \(f(\theta) = \sum p_k \theta^k\) be the associated pgf. Suppose that \(\rho \in (0, 1)\) satisfies \(f(\rho) = \rho\). Show that \(\rho^Z_n\) is a martingale, and use this to conclude that \(P(Z_n = 0 \text{ for some } n \geq 0) = \rho^x\).

9. Let \(X \geq 0, \, EX^2 < \infty\), and \(0 \leq a < EX\). Apply the Cauchy–Schwarz inequality to prove that \(P(X > a) \geq (EX - a)^2/EX^2\).

10. By considering the Poisson distribution, show that

\[
e^{-n} \left( 1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!} \right) \to \frac{1}{2}
\]

as \(n \to \infty\).