Probability Prelim

January 7, 2010

There are 10 problems, of which you should turn in solutions for exactly 6 (your best 6). Each problem is worth 10 points, and 40 points is required for passing. On the outside of your exam book, indicate which 6 you have attempted.

If you think a problem is misstated, interpret it in such a way as to make it nontrivial.

1. In this problem, all random variables are nonnegative. We say that $X$ is stochastically dominated by $Y$ if $P\{X > a\} \leq P\{Y > a\}$ for all $a > 0$. Prove that if $X$ is stochastically dominated by $Y$, then $E\Phi(X) \leq E\Phi(Y)$ for all increasing functions $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$.

2. Suppose $(\Omega, \mathcal{F}, P)$ is a probability space, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ defines a filtration of sigma-algebras of subsets of $\mathcal{F}$.

   (a) State, without proof, Doob’s martingale convergence theorem.

   (b) Prove that $\mathcal{L} := \lim_{n \to \infty} E(Z | \mathcal{F}_n)$ exists almost surely and in $L^1(P)$ for all $Z \in L^1(P)$ [this is Lévy’s martingale convergence theorem]. Identify the limit $\mathcal{L}$.

3. Suppose $X_1, X_2, \ldots$ are independent, identically-distributed exponential random variables with mean $\lambda > 0$. Prove that

   $$\max(X_1, \ldots, X_n) - \frac{1}{\lambda} \ln n \Rightarrow X,$$

   and compute $P\{X > x\}$ for all $x > 0$.

4. Give a rigorous proof that $E[XY] = E[X]E[Y]$ if $X$ and $Y$ are independent random variables belonging to $L^1(P)$. In particular, show that $XY \in L^1(P)$.

5. Fix $n \geq 2$ and let $X, Y_1, \ldots, Y_n$ be jointly distributed random variables. We say that $Y_1, \ldots, Y_n$ are conditionally i.i.d. given $X$ if

   $$P(Y_1 \leq y_1, \ldots, Y_n \leq y_n \mid X) = P(Y_1 \leq y_1 \mid X) \cdots P(Y_n \leq y_n \mid X)$$
for all \( y_1, \ldots, y_n \). Show that, if \( Y_1, \ldots, Y_n \) are conditionally i.i.d. given \( X \), then

\[
\text{Var}(Y_1 + \cdots + Y_n) = n^2 \text{Var}(Y_1) - n(n - 1)E[\text{Var}(Y_1 \mid X)].
\]

6. A random experiment has exactly three possible outcomes, referred to as outcomes 1, 2, and 3, with probabilities \( p_1 > 0 \), \( p_2 > 0 \), and \( p_3 > 0 \), where \( p_1 + p_2 + p_3 = 1 \). We consider a sequence of independent trials, at each of which the specified random experiment is performed. For \( i = 1, 2 \), let \( N_i \) be the number of trials needed for outcome \( i \) to occur, and put \( N := \min(N_1, N_2) \).

(a) Show that \( N \) is independent of \( 1_{\{N_1 < N_2\}} \).

(b) Evaluate \( E[N_1 \mid N_1 < N_2] \).

(c) Roll a pair of dice until a total of 6 or 7 appears. Given that 6 appears before 7, what is the (conditional) expected number of rolls?

7. If \( X \) is either (a) Poisson(\( \lambda \)) or (b) gamma(\( \lambda, 1 \)) (density proportional to \( x^{\lambda-1}e^{-x}, x > 0 \)), show that \( (X - E[X]) / \sqrt{\text{Var}(X)} \) converges in distribution to \( N(0, 1) \) as \( \lambda \to \infty \) (\( \lambda \) need not be an integer).

8. Let \( X_1, X_2, \ldots \) be i.i.d. with mean \( \mu \) and finite variance. Show that

\[
U_n := \left(\begin{pmatrix} n \\ 2 \end{pmatrix}\right)^{-1} \sum_{1 \leq i < j \leq n} X_i X_j
\]

converges in probability to \( \mu^2 \) as \( n \to \infty \).

9. Let \( X_1, X_2, \ldots \) be i.i.d. with mean \( \mu \). Define \( S_n := X_1 + \cdots + X_n \) for all \( n \geq 1 \). For fixed \( n \geq 2 \), define

\[
M_1 := \frac{S_n}{n}, \quad M_2 := \frac{S_{n-1}}{n-1}, \quad \ldots \quad M_n := \frac{X_1}{1}.
\]

(a) Show that \( E[X_k \mid S_n] = S_n/n \) for \( 1 \leq k \leq n \).

(b) Show that \( M_1, M_2, \ldots, M_n \) is a martingale.

10. Let \( Z \) be a random variable with all moments finite. Choose \( X \) and \( Y \) appropriately as in the Cauchy–Schwarz inequality or the Hölder inequality to prove that \( f(p) := \ln E[|Z|^p] \) is convex on \((0, \infty)\).