Probability Prelim

January 7, 2010

There are 10 problems, of which you should turn in solutions for **exactly** 6 (your best 6). Each problem is worth 10 points, and 40 points is required for passing. On the outside of your exam book, indicate which 6 you have attempted.

If you think a problem is misstated, interpret it in such a way as to make it nontrivial.

1. In this problem, all random variables are nonnegative. We say that X is *stochastically dominated by* Y if $\mathrm{P}\{X > a\} \leq \mathrm{P}\{Y > a\}$ for all $a > 0$. Prove that if X is stochastically dominated by Y, then $\mathrm{E}\Phi(X) \leq \mathrm{E}\Phi(Y)$ for all increasing functions $\Phi : \mathbf{R}_+ \to \mathbf{R}_+$.

2. Suppose $(\Omega, \mathcal{F}, \mathrm{P})$ is a probability space, and $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ defines a filtration of sigma-algebras of subsets of $\mathcal{F}$.

   (a) State, without proof, Doob's martingale convergence theorem.

   (b) Prove that $\mathcal{L} := \lim_{n\to\infty} \mathrm{E}(Z \,|\, \mathcal{F}_n)$ exists almost surely and in $L^1(\mathrm{P})$ for all $Z \in L^1(\mathrm{P})$ [this is *Lévy's martingale convergence theorem*]. Identify the limit $\mathcal{L}$.

3. Suppose $X_1, X_2, \ldots$ are independent, identically-distributed exponential random variables with mean $\lambda > 0$. Prove that

$$\max(X_1 \ \ldots, X_n) - \frac{1}{\lambda} \ln n \Rightarrow X,$$

   and compute $\mathrm{P}\{X > x\}$ for all $x > 0$.

4. Give a rigorous proof that $\mathrm{E}[XY] = \mathrm{E}[X]\mathrm{E}[Y]$ if X and Y are independent random variables belonging to $L^1(\mathrm{P})$. In particular, show that $XY \in L^1(\mathrm{P})$.

5. Fix $n \geq 2$ and let $X, Y_1, \ldots, Y_n$ be jointly distributed random variables. We say that $Y_1, \ldots, Y_n$ are *conditionally i.i.d. given* X if

$$\mathrm{P}(Y_1 \leq y_1, \ \ldots, \ Y_n \leq y_n \mid X) = \mathrm{P}(Y_1 \leq y_1 \mid X) \cdots \mathrm{P}(Y_1 \leq y_n \mid X)$$

for all $y_1, \dots, y_n$. Show that, if $Y_1, \dots, Y_n$ are conditionally i.i.d. given X, then

$$\operatorname{Var}(Y_1 + \cdots + Y_n) = n^2 \operatorname{Var}(Y_1) - n(n-1)\operatorname{E}[\operatorname{Var}(Y_1 \mid X)].$$

6. A random experiment has exactly three possible outcomes, referred to as outcomes 1, 2, and 3, with probabilities $p_1 > 0$, $p_2 > 0$, and $p_3 > 0$, where $p_1 + p_2 + p_3 = 1$. We consider a sequence of independent trials, at each of which the specified random experiment is performed. For $i = 1, 2$, let N_i be the number of trials needed for outcome i to occur, and put $N := \min(N_1, N_2)$.
   (a) Show that N is independent of $1_{\{N_1 < N_2\}}$.
   (b) Evaluate $\operatorname{E}[N_1 \mid N_1 < N_2]$.
   (c) Roll a pair of dice until a total of 6 or 7 appears. Given that 6 appears before 7, what is the (conditional) expected number of rolls?

7. If X is either (a) Poisson(λ) or (b) gamma($\lambda, 1$) (density proportional to $x^{\lambda-1}e^{-x}$, $x > 0$), show that $(X - E[X])/\sqrt{\operatorname{Var}(X)}$ converges in distribution to $N(0,1)$ as $\lambda \to \infty$ (λ need not be an integer).

8. Let $X_1, X_2, \dots$ be i.i.d. with mean μ and finite variance. Show that

$$U_n := \binom{n}{2}^{-1} \sum_{1 \le i < j \le n} X_i X_j$$

converges in probability to μ^2 as $n \to \infty$.

9. Let $X_1, X_2, \dots$ be i.i.d. with mean μ. Define $S_n := X_1 + \cdots + X_n$ for all $n \ge 1$. For fixed $n \ge 2$, define

$$M_1 := \frac{S_n}{n}, \quad M_2 := \frac{S_{n-1}}{n-1}, \quad \dots \quad M_n := \frac{X_1}{1}.$$

   (a) Show that $\operatorname{E}[X_k \mid S_n] = S_n/n$ for $1 \le k \le n$.
   (b) Show that $M_1, M_2, \dots, M_n$ is a martingale.

10. Let Z be a random variable with all moments finite. Choose X and Y appropriately as in the Cauchy–Schwarz inequality or the Hölder inequality to prove that $f(p) := \ln \operatorname{E}[|Z|^p]$ is convex on $(0, \infty)$.