Probability Prelim Exam

August 2014

Read the following instructions before you begin:

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam problems begin here:

1. Let $X_1, X_2, \ldots$ be independent random variables with 2 finite moments, and $\sum_{j=1}^{\infty} \mathrm{Var}(X_j) < \infty$. Prove that $\lim_{n\to\infty} \sum_{j=1}^{n}(X_j - \mathrm{E}X_j)$ exists a.s. and in $L^2(\mathrm{P})$.
2. Suppose $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are submartingales with respect to the same filtration $\{\mathscr{F}_n\}_{n=1}^{\infty}$. Prove that $Z_n := \max(X_n\,, Y_n)$ is also a submartingale.
3. Suppose $X = \mathrm{Poisson}(\lambda)$ for some $\lambda > 0$. Prove that $\mathrm{P}\{X > \mathrm{e}\lambda\} \leqslant \mathrm{e}^{-\lambda}$.
4. Suppose $\{X_j\}_{j=0}^{\infty}$ is a martingale such that $X_0 = 0$ and $|X_{i+1} - X_i| \leqslant 1$ for all $i \geqslant 0$. Prove that $\lim_{n\to\infty} n^{-\delta} X_n = 0$, a.s. for every $\delta > 1/2$.

5. Let $X_1, X_2, \ldots$ be independent, identically distributed random variables with the following common distribution:

$$\mathrm{P}\{X_1 = k\} = \frac{3}{\pi^2 k^2} \quad \text{for } k = \pm 1, \pm 2, \ldots .$$

(a) Prove that $\mathrm{E}\exp(itX_1) = 1 - (3|t|/\pi) + o(|t|)$ as $t \to 0$. You may use, without proof, the fact that $\int_0^\infty (1 - \cos\theta)\theta^{-2}\, \mathrm{d}\theta = \pi/2$.

(b) Use the preceding in order to prove that $n^{-1}(X_1 + \cdots + X_n)$ converges weakly to a "Cauchy random variable Y with scale parameter $3/\pi$." That is, the characteristic function of Y is

$$\mathrm{E}\exp(itY) = \exp(-3|t|/\pi) \qquad \text{for all } t \in \mathbf{R}.$$

You may use, without proof, the fact that the latter is a characteristic function. This matter is the topic of Problem 8 below.

6. Let $X_1, X_2, \ldots$ be i.i.d. integer-valued random variables with probability mass function $f(a) := \mathrm{P}\{X_1 = a\}$ for all $a \in \mathbf{Z}$, and suppose $f(a) > 0$ for all $a \in \mathbf{Z}$. Now let g be another probability mass function on $\mathbf{Z}$ and define the likelihood ratio,

$$\Lambda_n := \prod_{j=1}^n \frac{g(X_j)}{f(X_j)}.$$

(a) Consider the log-likelihood $\log \Lambda_n$, where "log" denotes the natural logarithm. Prove that $\lim_{n\to\infty} n^{-1} \log \Lambda_n = \mathbb{D}(g \parallel f)$ a.s., where

$$\mathbb{D}(g \parallel f) := \sum_{a=-\infty}^{\infty} f(a) \log\left[\frac{g(a)}{f(a)}\right],$$

provided that $\mathbb{E}(g \parallel f) := \sum_{a=-\infty}^{\infty} |\log[g(a)/f(a)]| f(a) < \infty$.

(b) Verify that $\log x \leqslant x - 1$ for all $x > 0$, and that the inequality is strict except when $x = 1$. Use this to prove the following [*Gibbs' inequality*]: $\mathbb{D}(g \parallel f) < 0$.

(c) Conclude from the preceding that if $\mathbb{E}(g \parallel f) < \infty$, then $\Lambda_n \to 0$ a.s. very rapidly as $n \to \infty$.

This fact plays an important role in statistics.

7. Suppose $f : [0\,,\infty) \to \mathbf{R}$ is bounded and measurable, and $f(x) = 0$ for all $x \geqslant 2$.

(a) Prove that if $X_1, X_2, \ldots$ are independent, all distributed uniformly in the interval $(0\,,1)$, then

$$\frac{1}{n}\sum_{j=0}^{n} f\left(\frac{j+X_j}{n}\right) \xrightarrow{\mathrm{P}} \int_0^\infty f(x)\,\mathrm{d}x \qquad \textit{as } n \to \infty.$$

(b) *(Extra credit)* Deduce from this the following version of Riemann sums for Lebesgue integrals [*Chaterjee's theorem*]: There exists a sequence $\{\delta_{j,n};\, 0 \leqslant j \leqslant n, n \geqslant 1\}$ of real numbers in $(0\,,1)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n} f\left(\frac{j+\delta_{j,n}}{n}\right) = \int_0^\infty f(x)\,\mathrm{d}x.$$

8. Let f be a continuous element of $L^1(\mathbf{R})$, and recall that its Fourier transform is $\hat{f}(t) := \int_{-\infty}^{\infty} \exp(itx) f(x)\,\mathrm{d}x$.

(a) Suppose also that $\hat{f} \in L^1(\mathbf{R})$. Prove the following [*inversion formula*]:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ixt}\hat{f}(t)\,\mathrm{d}t \qquad \text{for all } x \in \mathbf{R}.$$

In other words, $\hat{\hat{f}}(-x) = 2\pi f(x)$ for all $x \in \mathbf{R}$.

(b) Choose and fix some $\theta > 0$. Use the preceding to prove that the Fourier transform of the probability density function

$$f(x) := \frac{\theta}{\pi(\theta^2 + x^2)} \qquad [-\infty < x < \infty]$$

is $\hat{f}(t) = \exp(-\theta|t|)$.

9. Suppose X and Y are i.i.d. random variables and f and g are nondecreasing, bounded and measurable functions on $\mathbf{R}$.

   (a) Prove that $\mathrm{E}[(f(X) - f(Y))(g(X) - g(Y))] \geqslant 0$.

   (b) Conclude that $f(X)$ and $g(X)$ are always positively correlated; i.e., $\mathrm{E}[f(X)g(X)] \geqslant \mathrm{E}[f(X)] \cdot \mathrm{E}[g(X)]$.

10. Let $Z = \mathrm{N}(0\,,1)$ denote a standard normal random variable. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuously differentiable and there exists $c > 1$ such that $|f(x)| + |f'(x)| \leqslant c\exp(c|x|)$ for all $x \in \mathbf{R}$. Derive the following [*Stein's differential equation*]: $\mathrm{E}[f'(Z)] = \mathrm{E}[Zf(Z)]$.