Ph.D. Qualification Examination in Probabiity
January 2006

Correct and complete solutions to 5 problems guarantees a "pass."

1. If X is distributed uniformly on $[0\,,1]$, then compute $\mathrm{E}[X \mid \mathcal{G}]$, where $\mathcal{G}$ is the σ-algebra generated by $\{X \leq 1/2\}$.

2. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of i.i.d. random variables, and define

$$S(r) := \sum_{i=1}^{\infty} (1-r)^i X_i \qquad \text{for all } 0 < r < 1.$$

(a) Prove that for every fixed $r \in (0\,,1)$, $S(r)$ is a.s. absolutely convergent if and only if $\mathrm{E}\{\log(1+|X_1|)\} < \infty$.
(b) Suppose that $\mathrm{E}[X_1] = \mu$ and $\mathrm{Var}(X_1) = \sigma^2 < \infty$. Then prove that there exists a constant c such that $rS(r) \to c$ in probability, as $r \to 0$. Compute c.

3. Let $\{X_n\}_{n=1}^{\infty}$ denote a martingale such that $X_1 \in L^2(\mathrm{P})$, and

$$\sum_{n=1}^{\infty} \mathrm{E}\{(X_{n+1} - X_n)^2\} < \infty.$$

Prove that $\lim_{n\to\infty} X_n$ exists a.s. and in $L^2(\mathrm{P})$.

4. Let $X_1, X_2, \ldots$ be i.i.d., and $S_n := X_1 + \cdots + X_n$. Assume also that $\mathrm{P}\{X_1 = 1\} = \mathrm{P}\{X_1 = -1\} = 1/2$, so that $\{S_n\}_{n=1}^{\infty}$ is a simple symmetric random walk. Define

$$T := \inf\{k \geq 1 :\ |S_k| \geq 2\}.$$

As usual, $\inf \varnothing := \infty$. Prove that $\mathrm{E}[T^2] < \infty$.

5. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent (but not identically distributed) random variables, such that for all $n \geq 1$,

$$\mathrm{P}\{X_n = 1\} = \mathrm{P}\{X_n = -1\} = \frac{1}{2} - \frac{1}{2n^2}, \quad \mathrm{P}\{X_n = n\} = \mathrm{P}\{X_n = -n\} = \frac{1}{2n^2}.$$

Prove that $n^{-1/2}\sum_{i=1}^{n} X_i \Rightarrow N(0\,,\sigma^2)$, and compute σ^2 explicitly.

6. Suppose X and Y are two independent standard normal random variables.
(a) Prove that for all twice continuously differentiable functions $f, g : \mathbf{R} \to \mathbf{R}$,

$$\mathrm{Cov}\left(f(X), g(X)\right) = \int_0^1 \mathrm{E}\left[f'(X) g'\left(sX + (1-s^2)^{1/2} Y\right)\right] ds.$$

(`Hint`: Check it first for $f(x) := \exp(itx)$ and $g(x) := \exp(i\tau x)$.)

(b) Conclude the following "Poincaré inequality," due to J. Nash (1958):

$$\mathrm{Var}\,(f(X)) \le \mathrm{E}\left[(f'(X))^2\right].$$

7. Let $\{X_i\}_{i=1}^{\infty}$ denote a sequence of i.i.d. random variables with $\mathrm{P}\{X_1 = 0\} = p$ and $\mathrm{P}\{X_1 = 1\} = q := 1-p$. Consider the random sequence $X_1X_2\cdots$, and compute $\mathrm{E}[T]$, where T denotes the first time that the pattern "001" appears in the said sequence.

8. Construct an example of a sequence $\{A_n\}_{n=1}^{\infty}$ such that:

(a) (i) $\sum_n \mathrm{P}(A_n) = \infty$; and (ii) only finitely-many of the A_n's occur with positive probability.

(b) (i) $\sum_n \mathrm{P}(A_n) = \infty$; and (ii) infinitely-many of the A_n's occur with positive probability.

9. Let X and Y be two standard-normal random variables.

(a) Prove that if (X, Y) is Gaussian then X and Y are independent.

(b) Construct an example wherein (X, Y) is *not* a (two-dimensional) Gaussian random variable.

10. Let $\{X_i\}_{i=1}^{\infty}$ be i.i.d. with the "standard Cauchy distribution." That is, the density function of X_1 is

$$f(x) = \frac{1}{\pi(1+x^2)} \qquad -\infty < x < \infty.$$

Prove that $\max_{1\le i\le n} X_i/n$ converges weakly. Identify the limit.

Ph.D. Qualification Examination in Probabiity
January 2005

Correct solutions to 4 problems guarantees a "pass."

1. Prove that for all submartingales $\{X_n\}_{n=1}^{\infty}$,

$$\mathrm{P}\left\{\max_{n\le m}|X_n|\ge\lambda\right\}\le\frac{3}{\lambda}\max_{n\le m}\mathrm{E}\{|X_n|\}\qquad\lambda>0, m=1,2,\ldots.$$

2. Suppose $X_1, X_2, \ldots$ are independent with distribution,

$$\mathrm{P}\{X_n=n^{\alpha}\}=\mathrm{P}\{X_n=-n^{-\alpha}\}=\frac{1}{2}\qquad n=1,2,\ldots.$$

Prove that $\sum_{i=1}^{n}X_n/n\to 0$ a.s. if $0<\alpha<\frac{1}{2}$.

3. Suppose $X_{\alpha,\beta}$ has a Gamma distribution with parameters (α,β). That is, the density function of $X_{\alpha,\beta}$ is

$$f(x)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}\quad x>0,$$

where $\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}\,dx$ for all $\alpha>0$. Prove that $(\beta X_{\alpha,\beta}-\alpha)/\sqrt{\alpha}\Rightarrow N(0,1)$.

4. Let X be a mean-zero random variable with $\mathrm{E}(X^2)=\sigma^2<\infty$. Prove *Cantelli's inequality*: For all $\lambda>0$,

$$\mathrm{P}\{X\ge\lambda\}\le\inf_{t>0}\left[\frac{t^2+\sigma^2}{(t+\lambda)^2}\right]=\frac{\sigma^2}{\sigma^2+\lambda^2}.$$

Prove that this is a better bound than Chebyshev's inequality.

5. If $X\in L^2(\Omega,\mathcal{F},\mathrm{P})$, then prove that for all sub-σ-algebras $\mathcal{G}$ of $\mathcal{F}$,

$$\mathrm{Var}(X)=\mathrm{E}\left[\mathrm{Var}(X|\mathcal{F})\right]+\mathrm{Var}\left(\mathrm{E}\{X|\mathcal{F}\}\right)\qquad\text{a.s.}$$

6. Suppose $Y, X_1, X_2, \ldots$ have the following properties:

(a) Y has the exponential distribution. That is, $\mathrm{P}\{Y>t\}=e^{-t}$ for $t>0$.

(b) Conditionally on Y, $X_1,\ldots,X_n$ are i.i.d. exponentials with parameter Y; i.e., for all $t_1,\ldots,t_n>0$,

$$\mathrm{P}\left(X_1>t_1,\ldots,X_n>t_n\mid Y\right)=\prod_{i=1}^{n}e^{-Yt_i}=e^{-Y\sum_{i=1}^{n}t_i}\qquad\text{a.s.}$$

Compute $\mathrm{E}[Y \mid X_1, \ldots, X_n]$. Use this to prove that $\{M_n\}_{n=1}^{\infty}$ is a martingale, where

$$M_n = \frac{n+1}{1 + X_1 + \cdots + X_n} \qquad n = 1, 2, \ldots .$$

7. Suppose $X_1, X_2, \ldots$ have common mean μ and variances $\sigma_1^2, \sigma_2^2, \ldots$. Prove that if $\sup_n \sigma_n^2 < \infty$ and $\lim_{|i-j| \to \infty} \mathrm{E}[X_i X_j] = 0$, then $(X_1 + \cdots + X_n)/n$ converges to μ in probability.

8. Prove that whenever $X_n \Rightarrow X$ and $Y_n \xrightarrow{\mathrm{P}} c$ for a non-random $c > 0$, then $X_n / Y_n \Rightarrow c^{-1} X$. Use this to prove the following: If $\xi_1, \xi_2, \ldots$ are i.i.d. with $\mathrm{E}[\xi_1] = \mu$ and $\mathrm{Var}(\xi_1) = \sigma^2$ then

$$\frac{S_n - n\mu}{\sqrt{\sum_{i=1}^{n} (\xi_i - \overline{\xi}_n)^2}} \Rightarrow N(0,1).$$

where $S_n = \xi_1 + \cdots + \xi_n$, and $\overline{\xi}_n = S_n / n$.

9. Suppose $\{X_n\}_{n=1}^{\infty}$ is a martingale that is bounded in $L^1(\mathrm{P})$; i.e., $\sup_n \|X_n\|_1 < \infty$. Use Doob's decomposition to prove that $X_T \in L^1(\mathrm{P})$ for all stopping times T.

10. Suppose $\{X_n\}_{n=1}^{\infty}$ is a sequence of random variables that has the property that $\sup_n |X_n| \le 1$ a.s. Then use Doob's decomposition to prove that $\sum_n X_n$ converges a.s. iff $\sum_n \mathrm{E}[X_n \mid X_1, \ldots, X_{n-1}]$ converges a.s.

Ph.D. Qualification Examination in Probabiity
August 2004

Correct solutions to 3 problems guarantees a "pass."

1. A sequence $X_1, X_2, \ldots$ of random variables is said to be *uniformly integrable* if

$$\lim_{c\to\infty} \sup_{n\geq 1} \mathrm{E}\left\{|X_n| \; ; \; |X_n| > c\right\} = 0.$$

Prove that $X_n \to X$ in $L^1(\mathrm{P})$ if and only if both of the following happen: (i) $X_n \to X$ in probability; and (ii) $\{X_n\}_{n=1}^{\infty}$ is uniformly integrable.

2. Construct three random variables X, Y, and Z, such that any distinct pair of them are independent but (X, Y, Z) are not independent.

3. Suppose $\{X_n\}_{n=1}^{\infty}$ are i.i.d. standard normal random variables. Prove that $M_n := (n+1)^{-1/2} \exp(S_n^2/(2n+2))$ defines a martingale, where $S_n = \sum_{i=1}^{n} X_i$.

4. Choose and fix an integer $n \geq 1$. For all continuous functions $f : [0,1] \to \mathbf{R}$ define

$$B_n f(p) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} f(k/n), \qquad \text{for all } p \in [0,1].$$

[$B_n f$ is the Bernstein polynomial of f.] Prove that if f is increasing, then so is $B_n f$.

5. Let $\{X_n\}_{n=1}^{\infty}$ be i.i.d. random variables with distribution, $\mathrm{P}\{X_1 = 2\} = \mathrm{P}\{X_1 = -2\} = \frac{1}{2}$. Define $S_n = \sum_{i=1}^{n} X_i$ and $T = \inf\{n \geq 1 : S_n \in \{-2, 6\}\}$. Prove that $T < \infty$ a.s., and then compute $\mathrm{P}\{S_T = -2\}$.

6. Suppose $\{X_n\}_{n=1}^{\infty}$ and $\{Y_n\}_{n=1}^{\infty}$ are random variables on the same probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Suppose also that $X_n \Rightarrow X$ and $Y_n \Rightarrow Y$. True or false: $(X_n, Y_n) \Rightarrow (X, Y)$. [If true, then prove it. If false, then construct a counter-example.]

7. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables with $\mathrm{E}[X_1] = 0$ and $\mathrm{E}[X_1^2] = 1$. Define $S_k = X_1 + \cdots + X_k$ for all $k \geq 1$. Let N_m be another independent random variable with a mean-m Poisson distribution. Prove that there exist non-random α_m and β_m such that $(S_{N_m} - \alpha_m)/\beta_m$ converges weakly to a standard-normal distribution as $m \to \infty$. Decribe an explicit example of α_m and β_m.

8. If $X \geq 0$ and $X \in L^p(\mathrm{P})$ for all $p > 1$, then prove that $\lim_{p\to\infty} \|X\|_p$ exists.

9. Consider two random variables $X, Y \in L^2(\mathrm{P})$. Prove that if $\mathrm{E}[X \,|\, Y] = Y$ and $\mathrm{E}[X^2 \,|\, Y] = Y^2$, then $X = Y$ a.s.

10. Define $X_0 = 1$. Then iteratively define X_n so that for all Borel sets $A \subseteq \mathbf{R}$,

$$\mathrm{P}\left(X_n \in A \,\middle|\, X_0, \ldots, X_{n-1}\right) = \frac{\text{The lebesgue measure of } A \cap [0, X_{n-1}]}{X_{n-1}}.$$

Prove that $\lim_{n\to\infty} 2^n X_n = 0$ a.s. You may use—without proof—the following version of the Borel–Cantelli lemma [due to Paul Lévy]: If $\{\mathcal{F}_n\}_{n=1}^{\infty}$ is a filtration and A_n's are events such that $A_n \in \mathcal{F}_n$ and $\sum_{n=1}^{\infty} \mathrm{P}(A_n \,|\, \mathcal{F}_{n-1}) = \infty$ a.s., then $\mathrm{P}(A_n \text{ infinitely often }) = 1$.

Preliminary Examination 2000: Probability

Instructions: Choose 5 of the 8 problems, and write up solutions for these five problems *only*. 70 percent correct will be a passing score. This is a closed book exam.

1. Let $X_1, X_2, X_3, \ldots$ be independent and exponential with mean 1. Use the Borel–Cantelli lemmas to show that $\limsup_{n\to\infty} X_n/\log n = 1$ a.s.

2. Let X be such that $E[X^+] = \infty$ and $E[X^-] < \infty$, and let $X_1, X_2, X_3, \ldots$ be i.i.d. with common distribution that of X. With $S_n = X_1 + X_2 + \cdots + X_n$, show that $\lim_{n\to\infty} S_n/n = \infty$ a.s.

3. Let $X_1, X_2, X_3, \ldots$ be i.i.d. nonnegative random variables with mean 1 and variance 1, and put $S_n = X_1 + X_2 + \cdots + X_n$. Show that $2(\sqrt{S_n} - \sqrt{n})$ converges in distribution to a standard normal as $n \to \infty$.

4. Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables, each of which has all moments finite, and suppose that $\lim_{n\to\infty} E[(X_n)^k] = k!$ for each $k \geq 1$. Show that X_n converges in distribution.

5. Let $X_1, X_2, X_3, \ldots$ be i.i.d. with finite mean and variance, and put $S_n = X_1 + X_2 + \cdots + X_n$. Show that $S_n - nE[X_1]$ and $(S_n - nE[X_1])^2 - n\mathrm{Var}(X_1)$ are martingales.

6. Let $P\{X = 1\} = P\{X = -1\} = 1/2$, let $X_1, X_2, X_3, \ldots$ be i.i.d. with common distribution that of X, and put $S_n = X_1 + X_2 + \cdots + X_n$. Define $T = \min\{n \geq 0 : S_n = -A$ or $S_n = B\}$, where A and B are positive integers. Using the martingales of problem 5, find $P\{S_T = B\}$ and $E[T]$.

7. Let P be the transition matrix for a finite irreducible aperiodic Markov chain, and assume that P is doubly stochastic (row and column sums are 1). Find the limit $\lim_{n\to\infty} P^n_{ij}$ for all i and j, and provide justification.

8. Consider the Markov chain in the set of nonnegative integers with transitions $P(n, 0) = 1 - P(n, n+1) = p_n > 0$ for each $n \geq 0$. Give necessary and sufficient conditions on the sequence $\{p_n\}$ for the chain to be (a) positive recurrent, (b) null recurrent, and (c) transient.

Preliminary Examination 1998: Probability & Statistics

Instructions: You pass this exam if all the following conditions are satisfied.

(i) You got at least 15 points from probability

(ii) You got at least 15 points from statistics

(iii) You got at least 45 points total

Probability

1. Suppose $\{E_1, E_2, \cdots\}$ is a sequence of measurable (otherwise arbitrary) events. Suppose $\{F_1, F_2, \cdots\}$ is another sequence of measurable events also totally independent of all of the E_i's. Assume the following:

(a) $P(E_k,\text{ infinitely often }) = 1$;

(b) there exists some $p > 0$, such that for all $k \geq 1$, $P(F_k) \geq p$.

Prove: $P(E_k \cap F_k,\text{ infinitely often }) \geq p > 0$. (10 points)

2. Let $X_1, X_2, \dots$ be independent identically distributed positive random variables with $EX_1 = \mu$ and $\text{var}(X_1) = \sigma^2$. Let

$$N(t) = \min\{k : \sum_{1\leq i\leq k} X_i > t\}, \quad 0 < t < \infty.$$

(a) Show that

$$\lim_{t\to\infty} \frac{1}{t} N(t) = \frac{1}{\mu}$$

almost surely. (5 points)

(b) Show that $(N(t) - t/\mu)/(t\sigma^2/\mu^3)^{1/2}$ goes in distribution to a standard normal random variable. (5 points)

3. Let $X_1, X_2, \dots$ be independent random variables with distribution functions $F_1, F_2, F_3, \dots$. Let $Y = \sup_{1\leq i<\infty} X_i$.

(a) Show that $P\{Y < \infty\}$ is 0 or 1 depending on whether $\sum_{1\leq i<\infty}(1 - F_i(x))$ converges for some x. (5 points)

(b) Show that if $P\{Y < \infty\} = 1$, then $\prod_{1\leq i<\infty} F_i(x)$ converges for all x and it is the distribution function of Y.

4. Let X_1, X_2, X_3 be i.i.d., each with an exponential distribution with mean 1. Find the joint distribution of

$$Y_1 = \frac{X_1}{X_1 + X_2}, \qquad Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3}, \qquad Y_3 = X_1 + X_2 + X_3.$$

(7 points) In particular, are they independent? (3 points)

5. Suppose N is a positive random variable.

(a) Show that

$$P(N > 0) \geq \frac{(EN)^2}{EN^2}.$$

(5 points)

(b) Let $X_1, X_2, \cdots$ be i.i.d. random variables. Define $S_n = X_1 + \cdots + X_n$. Use part (a) to show that for any positive integer n,

$$P(S_j = 0, \text{ for some } 1 \le j \le n) \ge \frac{\sum_{j=1}^{n} P(S_j = 0)}{2\big(1 + \sum_{j=1}^{n} P(S_j = 0)\big)}.$$

(5 points)

Statistics

6.Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with distribution function F and density function f. We assume that the second derivative of f is bounded. We estimate f by

$$f_n(t) = \frac{F_n(t + h_n) - F_n(t - h_n)}{2h_n},$$

where F_n is the empirical distribution function and $h_n \to 0$ as $n \to \infty$.

(a) Show that $Ef_n(t) \to f(t)$ as $n \to \infty$. (5 points)

(b) Show that if $nh_n \to \infty$, then

$$\frac{f_n(t) - Ef_n(t)}{(var(f_n(t)))^{1/2}}$$

is asymptotically standard normal as $n \to \infty$. (5 points)

7. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a Poisson distribution with parameter $\theta > 0$.

(a) Show that $(-1)^{X_1}$ is unbiased for $e^{-2\theta}$, and use the Lehmann–Scheffé theorem to deduce a UMVUE of $e^{-2\theta}$ based on the sample. (Hint: The conditional distribution of X_1 given $X_1 + \cdots + X_n$ has a simple form.) (5 points)

(b) Argue that the UMVUE in part (a) is a consistent estimator of $e^{-2\theta}$. (5 points)

8. Let $X_1, \ldots, X_n$ be a random sample from the $N(\mu, \sigma^2)$ distribution, with both μ and σ^2 unknown.

(a) Derive the likelihood ratio test for $H : \sigma^2 = \sigma_0^2$ against all alternatives. Here σ_0^2 is a known positive constant. (5 points)

(b) Do the same for $H : \sigma^2 \ge \sigma_0^2$. (5 points)

9. Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, \theta)$ (i.e., normal with mean and variance both equal to θ), where $\theta > 0$.

(a) Give three pivotal quantities (or pivots) involving the entire sample, which respectively have a standard normal distribution, a t-distribution, and a chi-squared distribution, and indicate the numbers of degrees of freedom. (5 points)

(b) Use the normal pivotal quantity in part (a) to obtain a $100(1 - \alpha)$ percent confidence interval for θ. (5 points)

10. Let $X_1, X_2, \ldots, X_n$ be independent Poisson random variables with parameters $\theta_1, \theta_2, \ldots, \theta_n$. We wish to test $H_o: \theta_1 = \theta_2 = \ldots = \theta_n$ against the alternative that H_o is not true.

(a) Find the likelihood ratio test. (5 points)

(b) Show that the likelihood ratio is asymptotically normal (after centralizing and normalizing) under H_o. (5 points)

Preliminary Examination 1996: Probability & Statistics

Instructions: Choose 6 of the 10 problems, with at least two from probability and at least two from statistics, and write up solutions for these six problems *only*. 70 percent correct will be a passing score.

Probability

In Problems 1 and 2, $S_n = X_1 + \cdots + X_n$.

1. Prove Cantelli's theorem: If $X_1, X_2, \ldots$ are independent (but not necessarily identically distributed), mean zero, random variables with $\sup_n E[X_n^4] < \infty$, then $S_n/n \to 0$ a.s. as $n \to \infty$.

2. (a) Let $X_1, X_2, \ldots$ be i.i.d. Poisson random variables with mean 1, and let Z be $N(0,1)$. Denote $a^- = -\min\{a, 0\}$, and prove that

$$(*) \qquad E\left[\left(\frac{S_n - n}{\sqrt{n}}\right)^-\right] \to E[Z^-].$$

(b) Evaluate both expectations in (*) explicitly, and, noting the telescoping sum, deduce Stirling's formula for $n!$.

3. Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$, and let M be the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ consisting of the $\mathcal{G}$-measurable functions in L^2. Show that $T(X) = E[X \mid \mathcal{G}]$ coincides with the orthogonal projection of $L^2(\Omega, \mathcal{F}, P)$ onto M.

4. Consider the Markov chain in the state space $S = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with transitions $P(i, i+2) = p$ and $P(i, i-1) = 1 - p$, where $0 < p < 1$. Determine for which p this chain is recurrent and for which p it is transient.

5. Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion. Prove directly, using

$$V_n = \sum_{i=1}^{2^n} |B(i/2^n) - B((i-1)/2^n)|,$$

that $B(\cdot, \omega)$ is of unbounded variation on [0,1] for a.e. ω.

Statistics

6. Suppose $X_1, \ldots, X_n$ is an i.i.d. sample from a normal population with

$$EX_1 = \text{Var}(X_1) = \mu > 0.$$

(a) Compute the maximum likelihood estimator $\hat{\mu}$ of μ;
(b) Is $\hat{\mu}$ consistent?
(c) Is $\hat{\mu}$ asymptotically normal?

7. Suppose $X_1, \ldots, X_n$ is an i.i.d. sample from a normal distribution with mean μ and variance 1.
(a) Find the UMVU estimator for μ. (Prove the optimality criterion.)
(b) Put a $N(\theta, \tau^2)$ prior on μ and find the minimax estimator of μ.

8. Consider the linear model: $Y_i = \beta + \epsilon_i$, $1 \leq i \leq n$. Here, ϵ_i's are i.i.d. $N(0, \sigma^2)$, and σ and β are unknown.
(a) Find the least squares estimator of β;
(b) Find the UMVU estimator for β (prove the optimality);
(c) Find the UMP test for $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_1$;
(d) Discuss—without proofs—how to find the UMP test for $H_0 : \beta > \beta_0$ vs. $H_1 : \beta \leq \beta_0$ from tests of the form in part (c) above.

9. Let $X_1, \ldots, X_n$ be an i.i.d. sample which is uniformly chosen from the interval (θ_1, θ_2).
(a) Prove that there are no one-dimensional (i.e., not vector-valued) sufficient statistics for $h(\theta_1, \theta_2)$ where h is a one-to-one measurable function.
(b) Is there a one-dimensional (i.e., not vector-valued) sufficient statistics for $\mu = (\theta_1 + \theta_2)/2$?

10. Let $X_1, \ldots, X_n$ be an i.i.d. sample from a uniform $(0, \theta)$ distribution, where $\theta > 0$.
(a) Find the maximum likelihood estimator $\hat{\theta}$ for θ;
(b) Construct a $100(1 - \alpha)\%$ confidence interval for θ based on $\hat{\theta}$.

Preliminary Examination

PROBABILITY & STATISTICS

1994

You have 2 hours to complete this test.

Answer as many questions as you can. In order to insure a pass, you will need to solve as many as five questions total, with 1 complete solution in each subject.

This is an open book examination.

PROBABILITY QUESTIONS

1. Let $X_1, X_2, \cdots$ be independent, identically distributed random variables, uniformly distributed on $[0,1]$. Show that

$$Y_n = \frac{4\sum_{1\le k\le n} kX_k - n^2}{n^{3/2}},$$

converges in distribution to a normal random variable.

2. Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\{B(t), 0 \le t \le 1\}$ be a Brownian motion on it. Since almost all sample paths of B are continuous, $\int_0^t B(t)dt$ can be defined as a usual Rieman integral. Compute the distribution of $\int_0^1 B(t)dt$.

3. Let $\{S_n, \mathcal{F}_n,\ n \ge 1\}$ be a nonnegative martingale with $ES_n = 1$. Show that for all $\lambda > 0$,

$$P\big\{S_n \ge \lambda, \text{ for some } n \ge 1\big\} \le \frac{1}{\lambda}.$$

4. Let $X_1, X_2, \cdots$ be independent, identically distributed random variables. Show that the following statements are equivalent:

(a) $E|X_1|^\nu < \infty$;

(b) $X_n/n^{1/\nu} \to 0$, almost surely;

(c) $\max_{1\le i\le n} |X_i|/n^{1/\nu} \to 0$, almost surely.

5. Let $X_1, X_2, \cdots$ be independent, identically distributed normal random variables. Find two numerical sequences, a_n and b_n, such that

$$\frac{\max_{1\le i\le n} X_i - a_n}{b_n}$$

converges in distribution to a non–degenerate random variable.

6. Let $0 \le X_n \le 1$ be adapted to $\mathcal{F}_n$. Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and suppose

$$P\big(X_{n+1} = \alpha + \beta X_n \mid \mathcal{F}_n\big) = X_n$$
$$P\big(X_{n+1} = \beta X_n \mid \mathcal{F}_n\big) = 1 - X_n.$$

(a) Show that $P\{\lim_{n\to\infty} X_n = 0 \text{ or } 1\} = 1$.

(b) Show that if $X_0 = \theta$, then $P\{\lim_{n\to\infty} X_n = 1\} = \theta$.

1. Suppose $X_1, \cdots X_m$ are independent with $X_j \sim \text{Bin}(n_j, p_j)$.

(a) Find the UMVUE's of $p_1, \cdots p_m$.

(b) Suppose you know that $p_1 = p_2 = \cdots = p_m$. Let p denote this common (but unknown) value. Find the UMVUE of p.

(c) Find the likelihood ratio statistic, λ, for $H_0 : p_1, = \cdots = p_m$ versus $H_1 : p_i \neq p_j$, for some i and j.

(d) It can be shown that $2 \log \lambda$ is approximately the same as the usual χ^2 statistics. Using this fact, find an approximate test for H_0 vs H_1 above.

2. Consider the linear model:

$$Y_{ij} = \beta_i + \varepsilon_{ij}, \qquad 1 \leq i \leq 2, \qquad 1 \leq j \leq J.$$

Suppose ε_{ij}'s are independent and for some (known) a_1 and a_2, $\varepsilon_{ij} \sim \mathrm{N}(0, a_i^2\sigma^2)$.

(a) Find the U.M.V.U.E.'s of β_1 and β_2.

(b) Suppose you know that for some unknown β, $\beta_1 = \beta_2$. Find the U.M.V.U.E. of β.

3. Suppose $X \sim \text{Poiss}(\theta)$. Put a $\text{Gamma}(\alpha, \beta)$ prior on θ and suppose we have the following loss function: $\ell(\theta, a) = (a - \theta)^2/\theta$. Find the Bayes' estimator of θ.

4. Suppose $\theta \in \{\theta_0, \theta_1\}$ is an unknown (θ_0 and θ_1 are, however, known.) Put some prior, π, on θ. We are to test $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$. Our actions are 0 (accept) and 1 (reject). Find the Bayes' procedure for doing this test, if the loss function, $\ell(\theta, a)$, is the 0–1 loss given by:

$$\ell(0,1) = 0, \qquad \ell(0,0) = 1$$
$$\ell(1,0) = 0, \qquad \ell(1,1) = 1.$$

Is this procedure minimax?

5. Construct a $(1 - \alpha)$ two–sided confidence interval for the correlation coefficient of a bivariate normal distribution. (Hint. This is an exponential family.)

Some Densities

N(μ, σ^2) $\qquad x \in \mathbb{R}^1$: $\dfrac{1}{\sqrt{2\pi}\sigma} \exp\big(-\dfrac{1}{2\sigma^2}(x-\mu)^2\big).$

Gamma(α, β) $\qquad x \in \mathbb{R}^1$: $\dfrac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\int_0^\infty t^{\alpha-1} e^{-t} dt}.$

Poiss(λ) $\qquad x = 0, 1, \cdots$: $\dfrac{1}{x!} e^{-\lambda} \lambda^x.$

Bivariate Normal$(\mu_1, \mu_2, \rho, \sigma_1, \sigma_2)$ $\qquad (x, y) \in \mathbb{R}^2$:

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right\}\right).$$

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There are ten problems. Each counts 10 points. The minimum passing score is 60 points.

1. Let $X_0 = 1$ and define X_n inductively by declaring that X_{n+1} is uniformly distributed over $(0, X_n)$. Prove that $n^{-1} \log X_n \to c$ a.s. and compute c.

2. Let $X_1, X_2, \ldots$ be independent Poisson random variables with $EX_n = \lambda_n$ and let $S_n = X_1 + \cdots + X_n$. Show that if $\sum \lambda_n = \infty$, then $S_n / ES_n \to 1$ a.s.

   Hint: Show that (*) $Y_n / c_n \to 1$ a.s., provided that $Y_n \geq 0$ is nondecreasing in n, and (*) holds for a subsequence $n(k)$ that has $c_{n(k+1)}/c_{n(k)} \to 1$.

3. Let $X_1, X_2, \ldots$ be i.i.d. mean 0, variance $\sigma^2 \in (0, \infty)$.

   (a) Use the central limit theorem as well as Kolmogorov's 0–1 law to conclude that $\limsup S_n/\sqrt{n} = \infty$ a.s.

   (b) Show that $S_n/\sqrt{n}$ does not converge in probability.

4. Suppose that X and Y are independent. Let f be o Borel function on $\mathbf{R}^2$ with $E[|f(X,Y)|] < \infty$ and let $g(x) = E[f(x, Y)]$. Show that $E[f(X,Y) \mid X] = g(X)$.

5. A thinker who owns r umbrellas travels back and forth between home and office, taking along an umbrella (if there is one at hand) in rain (probability p) but not otherwise (probability $q = 1 - p$). Let the state be the number of umbrellas at hand, irrespective of whether the thinker is at home or at work. Set up the transition matrix, and show that the Markov chain approaches equilibrium (i.e., the ergodic theorem is applicable). Find the steady-state probability of his getting wet, and show that five umbrellas will protect him at the 5% level against any climate (any p).

6. Let $U_1, U_2, \ldots, U_n$ be independent identically distributed random variables, uniform on [0,1]. Let $U_{1,n} \le U_{2,n} \le \cdots \le U_{n,n}$ denote the order statistics. Let $X_1, X_2, \ldots, X_{n+1}$ be independent identically distributed exponential random variables with $EX_1 = 1$. Define $S(i) = X_1 + \cdots + X_i$.

   (a) Prove that the random vectors $\{U_{1,n}, \ldots, U_{n,n}\}$ and $\left\{\frac{S(1)}{S(n+1)}, \ldots, \frac{S(n)}{S(n+1)}\right\}$ have the same distribution.

   (b) Compute the asymptotic distribution of $n(U_{i+3,n} - U_{i,n})$, as $n \to \infty$ when i is fixed.

7. Let $Y_i = \alpha \frac{i}{n} + \varepsilon_i, \quad 1 \le i \le n$. We assume that $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent, identically distributed random variables with $E\varepsilon_i = 0$, $0 < \sigma^2 = var\ \varepsilon_i < \infty$ and $E\varepsilon_i^4 < \infty$.

   (a) Find the least-squares estimator for α.

   (b) Show that the estimator is asymptotically normal.

   (c) Find an estimator for σ^2.

8. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables, uniformly distributed on $[0, \theta]$, $\theta > \theta_0$. We want to test $H_0 : \theta = \theta_0$ against $H_A : \theta > \theta_0$.

   (a) Find the uniformly most powerful test. (You must prove your claim.)

   (b) Show that the uniformly most powerful test and the likelihood ratio test are equivalent.

   (c) Compute the power function of the most powerful test.

9. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with density function f. We assume that f' is bounded. Let K be a function satisfying $\int_{-\infty}^{\infty} K(u)du = 1$, $K'(u)$ is bounded and $K(u) = 0$, if $|u| \ge a$ where a is a constant. The density f is estimated by

$$\hat{f}_n(t) = \frac{1}{nh} \sum_{1 \le i \le n} K\left(\frac{t - X_i}{h}\right).$$

Show that $\hat{f}_n(t)$ is an almost surely uniformly consistent estimator for f on $[\alpha, \beta]$, $-\infty < \alpha < \beta < \infty$.

10. Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with $P\{X_i = 1\} = p, \quad P\{X_i = 0\} = 1 - p$.

    (a) Compute the maximum likelihood estimator of $\sigma^2 = p(1-p)$.

    (b) Compute the bias, the variance and the mean-square error of the estimator.

    (c) Is the estimator asymptotically efficient?

    (d) Find the uniformly minimum variance unbiased estimator for σ^2.