Exam Problems:

1. Consider an array \( \{X_{i,j}\} \) of random variables such that, for every \( n \geq 1 \), the collection \( X_{1,n}, \ldots, X_{n,n} \) is i.i.d. with common distribution

\[
P\{X_{1,n} = 1\} = P\{X_{1,n} = -1\} = \frac{1}{2} \left( 1 - \frac{1}{n^2} \right);
\]

and \( P\{X_{1,n} = n^2\} = n^{-2} \). Let \( S_n := X_{1,n} + \cdots + X_{n,n} \). Prove that \( S_n/\sqrt{n} \Rightarrow N(0, \sigma^2) \) for some \( \sigma \in (0, \infty) \), although \( E(S_n/\sqrt{n}) \rightarrow \infty \) as \( n \rightarrow \infty \).

2. Consider a one-dimensional simple symmetric random walk: \( S_n = x + \sum_{k=1}^{n} X_k \), where \( x \) is where the walk starts and \( \{X_n : n \geq 1\} \) are i.i.d. with

\[
P(X_n = 1) = P(X_n = -1) = 1/2.
\]

(a) Compute the probability that starting at \( x = 0 \) the random walk will hit \(-1\) before it hits \( b \in \mathbb{N} \). Deduce from this that the random walk is recurrent, i.e. starting at 0 the random walk will return to 0 with probability 1.

(b) Compute the mean time it takes the random walk to go from \( x = 0 \) to \(-1\). Deduce from this that the random walk is null recurrent, i.e. starting at 0, the mean time it takes to return to 0 is infinite.

(Hint: Prove first that when \( x = 0 \) both \( S_n \) and \( S_n^2 - n \) are martingales.)
3. Let $X_1, X_2, \ldots$ be a collection of i.i.d. random variables, and define $S_n := X_1 + \cdots + X_n$ for all $n \geq 1$. Suppose 
$$P\{X_1 = k\} = \frac{3}{\pi^2 k^2} \quad \text{for all } k = \pm 1, \pm 2, \ldots.$$ 
Prove that $S_n/n$ converges weakly to a non-constant random variable as $n \to \infty$.

4. Suppose $X, Y \in L^1(P)$ satisfy $E[X|Y] = Y$ and $E[Y|X] = X$ almost surely. Prove that $X = Y$ almost surely. (Hint: Prove first that $E[X - Y; X < q < Y] = 0$ for all $q$.)

5. Let $p_n$ and $q_n$ denote two probability mass functions on $\mathbb{Z}^n$, for every integer $n \geq 1$. That is, for all $n \geq 1$:
   
   (a) $p_n(x) \geq 0$ and $q_n(x) \geq 0$ for all $x \in \mathbb{Z}^n$; and
   
   (b) $\sum_{x \in \mathbb{Z}^n} p_n(x) = \sum_{x \in \mathbb{Z}^n} q_n(x) = 1$.

Now suppose $\{X_i\}_{i=1}^\infty$ is a collection of random variables such that $(X_1, \ldots, X_n)$ has mass function $p_n$ for every $n \geq 1$. That is, 
$$P\{X_1 = x_1, \ldots, X_n = x_n\} = p_n(x_1, \ldots, x_n) \quad \text{for all } (x_1, \ldots, x_n) \in \mathbb{Z}^n.$$

(a) Prove that $p_n(X_1, \ldots, X_n) > 0$ a.s. for all $n \geq 1$.

(b) Prove that $\{Z_n\}_{n=1}^\infty$ is a martingale, where, for all $n \geq 1$,

$$Z_n := \frac{q_n(X_1, \ldots, X_n)}{p_n(X_1, \ldots, X_n)},$$

almost surely on the P-measure-one event $\{p_n(X_1, \ldots, X_n) > 0\}$, and $Z_n := 0$ otherwise.

(c) Prove that $\lim_{n \to \infty} Z_n$ exists a.s. and is finite a.s. (This is a starting point for likelihood-ratio testing in classical statistics.)

6. If $X_1, X_2, \ldots$ are independent standard normal random variables, then find non-random sequences $a_n, b_n \to \infty$ such that $a_n \min_{1 \leq i \leq n} X_i + b_n$ converges weakly. Identify the limiting distribution.

7. Suppose $X_1, X_2, \ldots$ are independent mean-zero random variables that satisfy the condition $\sum_{i=1}^\infty \text{Var}(X_i) < \infty$.

(a) Prove that $\sum_{n=1}^\infty X_n$ exists a.s. and is finite a.s.

(b) Use part (a) to prove that if $\varepsilon_1, \varepsilon_2, \ldots$ are i.i.d. with $P\{\varepsilon_1 = \pm 1\} = 1/2$, then the random harmonic series $\sum_{i=1}^\infty (\varepsilon_i/i)$ converges a.s. and is finite a.s.

(c) Is $\sum_{i=1}^\infty (\varepsilon_i/i)$ in $L^1(P)$?

8. Suppose that $X_1, X_2, \ldots$ are i.i.d. exponential random variables with parameter $\lambda > 0$. Prove that $\limsup_{n \to \infty} (X_n / \log n) = \lambda^{-1}$ and $\liminf_{n \to \infty} (\log X_n / \log n) = -1.$
9. Let $X_1, X_2, \ldots$ be i.i.d. with $P\{X_1 = 1\} = p$ and $P\{X_1 = -1\} = q$, where $p \notin \{0, \frac{1}{2}, 1\}$ and $q = 1 - p$. Define $S_n := X_1 + \cdots + X_n$ for all $n \geq 1$ and consider the stopping time, $T := \inf\{n \geq 1 : |S_n| = 5\}$, where $\inf\emptyset := \infty$. Prove that $T \in L^1(P)$ and compute $E(T)$. (HINT: Begin by proving that $M_n := \left(\frac{q}{p}\right)^{S_n}$ and $N_n := S_n - (p - q)n$ are martingales.)

10. Let $X$ and $Y$ be two real-valued random variables. Let $f$ and $g$ be two non-decreasing measurable functions such that $f(X)$ and $g(Y)$ are integrable. Prove that

$$E[f(X)g(Y)] \geq E[f(X)]E[g(Y)].$$