Direction. Turn in solutions for no more than 6 of the 10 problems. Each is worth 10 points. 40 points are required to pass. If a problem seems to be misstated, interpret it so as to be nontrivial.

1. If $X \in L^2$, then it is easy to show that $\mu$ is its mean if and only if $E[(X - a)^2]$ is uniquely minimized at $a = \mu$. Assuming $X \in L^1$, show that $m$ is a median of $X$ if and only if $E[|X - a|]$ is minimized (not necessarily uniquely) at $a = m$.

2. (a) Find a simple asymptotic formula for $P(X > x)$, where $X$ has the $\text{GAMMA}(\theta, \alpha)$ density,

$$f(x) = \frac{1}{\theta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\theta}, \quad x > 0.$$

(b) Let $X_1, X_2, \ldots$ be i.i.d. $\text{GAMMA}(\theta, \alpha)$. Find an increasing sequence $a_n$ such that $\limsup_{n \to \infty} X_n/a_n = 1$ a.s., and prove it.

3. (a) Find the distribution of $\int_0^1 B_t \, dt$, where $(B_t)$ is Brownian motion.

(b) Given an i.i.d. sequence $X_1, X_2, \ldots$ with mean 0 and variance 1, what does Donsker's invariance principle tell us about this sequence in connection with the distribution of $\int_0^1 B_t \, dt$?

4. Let $X_1, X_2, \ldots$ be i.i.d. with common distribution $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$, and put $S_0 := 0$ and $S_n := X_1 + \cdots + X_n$ for each $n \geq 1$. Consider the stopping time $N := \min\{n \geq 1 : S_n = 1\}$.

(a) Let $g(u) := E[u^{S_n}]$ for all $u > 1$, and show that the optional stopping theorem applies to the martingale $M_n := u^{S_n} g(u)^{-n}$, where $u > 1$, and the stopping time $N$.

(b) Use part (a) to show that

$$E[v^N] = \frac{1 - \sqrt{1 - v^2}}{v}, \quad 0 < v < 1.$$ 

(c) Use part (b) to show that

$$P(N = 2m + 1) = \frac{1}{m+1} \binom{2m}{m} \frac{1}{2^{2m+1}}, \quad m \geq 0.$$ 

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5. Suppose \( g \) and \( h \) are continuous on \( \mathbb{R} \) with \( g > 0 \) and \( |h(x)|/g(x) \to 0 \) as \( |x| \to \infty \). If \( X_n \Rightarrow X \) and \( \sup_{n} E[g(X_n)] < \infty \), show that \( E[h(X)] = E[h(X)] \).

6. Let \( U_0, U_1, \ldots \) be independent and identically distributed random variables, uniform on \([0,1] \). Let \( N \) be a Poisson random variable with \( E[N] = 1 \), independent of \( U_0, U_1, \ldots \). Compute \( E[Y] \), where \( Y = \max_{0 \leq k \leq N} U_k \).

7. Let \( X_1, X_2, \ldots \) be independent and identically distributed random variables uniform on \([0,1] \) and let \( \alpha > 0 \). Show that there are numerical sequences\( a_n \) and \( b_n \) such that
\[
Y_n = \sum_{k=1}^{n} k^\alpha X_k - a_n
\]
converges in distribution to a standard normal random variable.

8. Let \( \{e_i, -\infty < i < \infty \} \) be independent and identically distributed random variables with \( E[e_1] = 0 \) and \( E[e_i^2] = \sigma^2 \) and \( \rho \) be a real number satisfying \(|\rho| < 1\). Show that
\[
Y = \sum_{i=0}^{\infty} \rho^i e_i
\]
is finite with probability one. Compute \( E[Y] \) and \( E[Y^2] \). (You need to justify your steps when you compute these two expected values.)

9. Compute
\[
\lim_{n \to \infty} (b-a)^n \int_a^b \int_a^b \cdots \int_a^b \frac{x_1 + x_2 + \cdots + x_n}{x_1^2 + x_2^2 + \cdots + x_n^2} \, dx_1 \, dx_2 \cdots dx_n
\]
if \( 0 < a < b < \infty \).

10. Let \( X_1, X_2, \ldots \) be independent and identically distributed random variables with characteristic function \( \varphi \). Let \( N \) be a random variable with
\[
P(N = k) = \frac{1}{2^k}, \quad k = 1, 2, \ldots
\]
We assume that \( \{X_i, i \geq 1\} \) and \( N \) are independent.

(a) Compute the characteristic function of \( Y = X_1 + X_2 + \cdots + X_N \).
(b) Can you weaken the condition that \( \{X_i, i \geq 1\} \) and \( N \) are independent and the formula obtained in (a) remains valid?