Probability Prelim Exam

August 2018

Instructions (Read before you begin)
- You may attempt all of 10 problems in this exam. However, you can turn in solutions for at most 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Let $\mu$ be a translation invariant $\sigma$-additive measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\mu((0, 1]) < \infty$. Prove that there exists a $c \in (0, \infty)$ such that $c\mu$ is Lebesgue measure.

2. Let $\{X_i : i \geq 1\}$ be a sequence of non-negative, i.i.d. random variables such $E[X_1] = \infty$. Prove that
   $$\lim_{n \to \infty} \frac{X_1 + \cdots + X_n}{n} = \infty \text{ almost surely.}$$

3. Let $\mu$ and $\nu$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $\lambda = \frac{1}{2}\mu + \frac{1}{2}\nu$. Let $\mathcal{F}$ be a $\sigma$-algebra contained in $\mathcal{B}(\mathbb{R})$. Let $\mu_\mathcal{F}$, $\nu_\mathcal{F}$, and $\lambda_\mathcal{F}$ be the restrictions of $\mu$, $\nu$, and $\lambda$ to $\mathcal{F}$, respectively.
   a) Prove that $\mu_\mathcal{F} \ll \lambda_\mathcal{F}$ and $\nu_\mathcal{F} \ll \lambda_\mathcal{F}$.
   b) What is $f_\mu = \frac{d\mu_\mathcal{F}}{d\lambda_\mathcal{F}}$ in terms of $\frac{d\mu}{d\lambda}$? Of course, a similar relationship holds for $f_\nu = \frac{d\nu_\mathcal{F}}{d\lambda_\mathcal{F}}$.
   c) For a bounded $\mathcal{B}(\mathbb{R})$-measurable function $g$, compute $E^\lambda[g \mid \mathcal{F}]$ in terms of $E^\mu[g \mid \mathcal{F}]$, $E^\nu[g \mid \mathcal{F}]$, $f_\mu$, and $f_\nu$?

4. Let $\{X_i : i \geq 1\}$ be i.i.d. integrable random variables. Prove that
   $$\lim_{n \to \infty} \frac{1}{\log n} \sum_{i=1}^{n} \frac{X_i}{i} = E[X_1].$$
5. Let \( \{X_i : i \geq 1\} \) be i.i.d. random variables distributed uniformly on \([0, 1]\). Prove that the distribution of
\[
\frac{4 \sum_{i=1}^{n} iX_i - n^2}{n^{3/2}}
\]
converges weakly and identify the limiting distribution.

6. Let \( \{X_n : n \geq 1\} \) be a stochastic process adapted to a filtration \( \{\mathcal{F}_n : n \geq 1\} \) and satisfying \( \mathbb{E}[|X_n|] < \infty \) for all \( n \geq 1 \). Prove that \( \{X_n : n \geq 1\} \) is a martingale if and only if \( \mathbb{E}[X_T] = \mathbb{E}[X_1] \) for all bounded stopping times \( T \).

7. Let \( \{X_n : n \geq 1\} \) be a martingale with respect to filtration \( \{\mathcal{F}_n : n \geq 1\} \), and let \( T \) be a stopping time with respect to \( \{\mathcal{F}_n : n \geq 1\} \). Prove the following two forms of the optional stopping theorem:
   a) If \( T \) is almost surely bounded then \( \mathbb{E}[X_T] = \mathbb{E}[X_1] \),
   b) If \( \mathbb{E}[\max_{1 \leq n \leq T} |X_n|] < \infty \), then \( \mathbb{E}[X_T] = \mathbb{E}[X_1] \).

8. Prove the identity
\[
\frac{\sin \theta}{\theta} = \int_0^1 e^{i \theta (2x-1)} \, dx.
\]
Then use this to show that
\[
\frac{\sin \theta}{\theta} = \prod_{n=1}^{\infty} \cos(\theta/2^n).
\]
**Hint:** Use that a Uniform(0, 1) random variable can be written as an infinite weighted sum of iid Bernoulli(1/2) random variables.

9. Let \( Z_1, Z_2, \ldots \) be iid \( N(0, 1) \) random variables. First prove Mills’ ratio
\[
\mathbb{P}(Z_i > \lambda) \leq \frac{1}{\lambda \sqrt{2\pi}} e^{-\lambda^2/2}
\]
for any \( \lambda > 0 \). Use this to show that for any \( \epsilon > 0 \)
\[
\mathbb{P} \left( \limsup_{n \to \infty} \frac{\max_{k \leq n} Z_k}{\sqrt{(2-\epsilon) \log n}} > 1 \right) = 1.
\]

10. Show that for every \( \rho \in (-1, 1) \) the function
\[
f(x, y) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \exp \left( -\frac{x^2 - 2\rho xy + y^2}{2(1 - \rho^2)} \right)
\]
is a pdf on \( \mathbb{R}^2 \), and that if \( (X, Y) \) is a pair of random variables with pdf \( f \) then
\[
\mathbb{P}(XY < 0) = \frac{1}{\pi} \arccos \rho.
\]