Preliminary Exam, Numerical Analysis, January 2009

Instructions: This exam is closed books and notes, and no electronic devices are allowed. The allotted time is three hours and you need to work on any three out of questions 1-4 and any two out of questions 5-7. All questions have equal weight and a score of 75 % is considered a pass. Indicate clearly the work that you wish to be graded.

-1- (Polynomial Interpolation.) Let $I = [a, b]$ , let $x_i$, $i = 0, \ldots, n$, be $n+1$ distinct points in $I$, and let $y_i$, $i = 0, \ldots, n$, be $n+1$ given real numbers. Show that there exists a unique polynomial

$$p(x) = \sum_{i=0}^{n} \alpha_i x^i$$

such that

$$p(x_i) = y_i, \quad i = 0, \ldots, n.$$  

-2- (Error Analysis.) Consider the linear system

$$Ax = b$$  

where $A$ is an invertible matrix. Let $\hat{x}$ be an approximation of the solution of (1) and let $e = x - \hat{x}$ and $r = b - A\hat{x}$.

Let $\| \cdot \|$ denote any vector norm or the corresponding induced matrix norm. Show that

$$\frac{\|e\|}{\|x\|} \leq \|A\|\|A^{-1}\| \frac{\|r\|}{\|b\|}.$$  

Comment on the significance of the condition number $\|A\|\|A^{-1}\|$ and give a lower bound for it in terms of the eigenvalues of $A$.

-3- (Linear Programming.) Define the phrase “Linear Programming Problem”. Let $A$ be a given $m \times n$ matrix with $m > n$, and let $b \in \mathbb{R}^m$ be a given vector. Write the problem

$$\text{Find } x \in \mathbb{R}^n \text{ such that } \|Ax - b\|_\infty = \min$$

as a linear programming problem.

-4- (The Gershgorin Theorem.) Let $\lambda$ be an eigenvalue of $A \in A^{n \times n}$. Show that there exists an

$$i \in \{1, 2, \ldots, n\}$$  

such that

$$|a_{ii} - \lambda| \leq \sum_{\substack{j=1 \atop i \neq j}}^{n} |a_{ij}|.$$
For every eigenvalue \( \lambda \), the inequality (3) describes a circle in the complex plane called a Gershgorin circle. Let \( S \) be a set that is the union of \( k \leq n \) Gershgorin circles such that the intersection of \( S \) with all other Gershgorin circles is empty. Show that \( S \) contains precisely \( k \) eigenvalues of \( A \) (counting multiplicities). Without proof or counterproof state whether it is possible for a Gershgorin Circle not to contain any eigenvalue at all.

-5- **(Adaptive Quadrature.)** Describe the basic idea of adaptive quadrature, and give a simple example, including formulas.

-6- **(Linear Multistep Methods.)** Consider the initial value problem

\[
y' = f(x, y), \quad y(a) = y_0
\]

Let \( h \) be some stepsize, \( x_n = a + nh, y_n \approx y(x_n) \), and \( f_n = f(x_n, y_n) \), for \( n = 0, 1, 2, 3, \ldots \). Let \( k \) be some step number, and ignore the question of obtaining starting values \( y_1, y_2, \ldots, y_{k-1} \). Suppose the approximations \( y_n, n = k, k+1, \ldots \) are obtained by the Linear Multistep Method

\[
\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}.
\]

Define what is meant by the **local truncation error** and the **order** of the linear multistep method (4). Compute the order and the local truncation error of Euler’s Method

\[
y_{n+1} - y_n = hf_n.
\]

-7- **(Numerical PDEs.)** Consider the one-dimensional heat equation: Find \( u(x, t) \) such that

\[
u_t = u_{xx}, \quad t \geq 0, \quad x \in [0, 1], \quad u(x, 0) = f(x), \quad u(0, t) = u(1, t) = 0.
\]

Describe how this problem might be solved by applying the Method of Lines and Euler’s Method. Give formulas that could be used to write a suitable computer code.