Preliminary Examination, Numerical Analysis, January 2017

Instructions: This exam is closed books and notes. The time allowed is three hours and you need to work on any three out of questions 1-5 and any two out of questions 6-8. All questions have equal weights and the passing score will be determined after all the exams are graded. Indicate clearly the work that you wish to be graded.

Note: In problems 6-8, the notations $k = \Delta t$ and $h = \Delta x$ are used.

1. Singular Value Decomposition (SVD):

   a) Prove the following statement:

   Singular Value Decomposition: Any matrix $A \in \mathbb{C}^{m \times n}$ can be factored as $A = U \Sigma V^*$, where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary and $\Sigma \in \mathbb{R}^{m \times n}$ is a rectangular matrix whose only nonzero entries are non-negative entries on its diagonal.

   b) Relate the matrices $U$, $V$, and $\Sigma$ to the four fundamental subspaces associated with $A$, that is, the range and null spaces of $A$ and $A^T$.

2. Linear Least Squares:

   The Linear Least Squares problem for an $m \times n$ real matrix $A$ and $b \in \mathbb{R}^m$ is the problem:
   
   Find $x \in \mathbb{R}^n$ such that $\|Ax - b\|_2$ is minimized.

   a) Suppose that you have data $\{(t_j, y_j)\}, j = 1, 2, \ldots, m$ that you wish to approximate by an expansion
   
   $p(t) = \sum_{k=1}^{n} x_k \phi_k(t)$.
   
   Here, the functions $\phi_k(t)$ are given functions. Which norm on the difference between the approximation function $p$ and the data gives rise to a linear least squares problem for the unknown expansion coefficients $x_k$? What is the matrix $A$ in this case, and what is the vector $b$?

   b) Suppose that $A$ is a real $m \times n$ matrix of full rank and let $b \in \mathbb{R}^m$. What are the ‘normal equations’ for the Least Squares problem? How can they be used to solve the Least Squares problem? What is the $QR$ factorization of $A$ and how can it be used to solve the Least Squares problem? Compare and contrast these approaches for numerically solving the Least Square problem.
3. Sensitivity:

a) Suppose that $A$ is an $n \times n$ nonsingular real matrix. Analyze the sensitivity of solutions of the system $Ax = b$ to perturbations in $b$. What quantity related to $A$ characterizes this sensitivity?

b) Suppose $\tilde{x}$ is an approximate solution to the linear system $Ax = b$, where $A$ is an $n \times n$ nonsingular real matrix. The residual is the vector $r = b - A\tilde{x}$. Derive inequalities relating the residual $r$ to the error $e = x - \tilde{x}$.

4. Interpolation and Integration:

a) Consider $n + 1$ distinct points $x_0 < x_1 < \ldots < x_n$ in the interval $[a, b]$. Let $f(x)$ be a smooth function defined on $[a, b]$. Show that there is a unique polynomial $p(x)$ of degree $n$ which interpolates $f$ at all of the points $x_j$. Derive the formula for the interpolation error at an arbitrary point $x$ in the interval $[a, b]$:

$$f(x) - p(x) \equiv E(x) = \frac{1}{(n+1)!} (x-x_0)(x-x_1) \cdots (x-x_n) f^{n+1}(\eta).$$

for some $\eta \in [a, b]$.

b) Consider the problem of approximating the integral $I(f) = \int_{a}^{b} f(x) dx$ by a formula of the type $I_n(f) = \sum_{j=1}^{n} a_j f(x_j)$ where $x_1, x_2, \ldots, x_n$ are distinct points in the interval $(a, b)$. Derive formulas for $a_j, j = 1, \ldots, n$ so the $I_n(f) = I(f)$ when $f$ is any polynomial of degree less than or equal to $n$.

c) For the same approximate integration problem as in (b), explain how to choose the points $x_1, x_2, \ldots, x_n$ and coefficients $a_j, j = 1, \ldots, n$, so that $I_n(f) = I(f)$ for all polynomials of degree less than or equal to $2n - 1$? Prove that your proposed choice does give the exact integral for these polynomials.
5. Iterative Methods:

Consider the fixed-point iteration

\[ u^{(k+1)} = Tu^{(k)} + c \]

for finding a solution of the problem

\[ u = Tu + c, \]

where \( T \) is an \( m \times m \) real matrix and \( c \) is a real \( m \)-vector.

a) Show that the fixed point iteration will converge for an arbitrary initial guess \( u^{(0)} \) if and only if the spectral radius of \( T \), \( \rho(T) \), is less than 1.

b) Consider the boundary value problem

\[-u''(x) = f(x), \quad \text{for} \quad 0 \leq x \leq 1\]

with \( u(0) = u(1) = 0 \), and the following discretization of it:

\[-U_{j-1} + 2U_j - U_{j+1} = F_j, \]

for \( j = 1, 2, \ldots, N-1 \) where \( Nh = 1 \) and \( F_j \equiv h^2 f(jh) \).

Show that the Jacobi iterative method will converge for this problem for any choice of initial guess. Express the speed of convergence as a function of the discretization stepsize \( h \). How does the number of iterations required to reduce the initial error by a factor \( \delta \) depend on \( h \)? In practice, would you use this method to solve the given problem? If so, explain why this is a good idea? If not, how would you solve it in practice?

6. Elliptic Problems:

For the one dimensional Poisson problem for \( v(x) \)

\[-v''(x) + \alpha v(x) = f(x), \]

where \( \alpha \geq 0 \) is constant, along with Dirichlet boundary conditions in the interval \([0,1]\), consider the scheme

\[ \Delta_h U_j \equiv \frac{1}{h^2} \left( -U_{j-1} + 2U_j - U_{j+1} \right) = f_j \]

for \( j = 1, 2, \ldots, N-1 \) where \( Nh = 1 \), \( f_j \equiv f(jh) \), and \( U_0 = U_N = 0 \). The approximate solution satisfies a linear system \( AU = b \), where \( U = (U_1, U_2, ..., U_{N-1})^T \) and \( b = h^2 (f_1, f_2, ..., f_{N-1})^T \).

a) State and prove the maximum principle for any grid function \( V = \{V_j\} \) with values for \( j = 0, 1, \ldots, N \), that satisfies \( \Delta_h V_j \geq 0 \).

b) Derive the matrix \( A \) and show that it is symmetric and positive definite.

c) Use the maximum principle to show that the global error \( e_j = v(x_j) - U_j \) satisfies \( \|e\|_\infty = O(h^2) \) as the space step \( h \to 0 \).
7. Numerical Methods for ODEs:

Consider the Linear Multistep Method

\[
y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}kf_{n+2}
\]

for solving an initial value problem \( y' = f(y, x), \ y(0) = \eta \). You may assume that \( f \) is Lipschitz continuous with respect to \( y \) uniformly for all \( x \).

a) Analyze the consistency, stability, accuracy, and convergence properties of this method.

b) Sketch a graph of the solution to the following initial value problem.

\[
y' = -10^8[y - \cos(x)] - \sin(x), \quad y(0) = 2.
\]

Would it be more reasonable to use this method or the forward Euler method for this problem? What would you consider in choosing a timestep \( k \) for each of the methods? Justify your answer.

8. Heat Equation Stability:

Consider the variable coefficient diffusion equation

\[
v_t = (\beta(x)v_x)_x, \quad 0 < x < 1, \ t > 0
\]

with Dirichlet boundary conditions

\[
v(0, t) = 0, \quad v(1, t) = 0
\]

and initial data \( v(x, 0) = f(x) \). Assume that \( \beta(x) \geq \beta_0 > 0 \), and that \( \beta(x) \) is smooth. Let \( \beta_{j+1/2} = \beta(x_{j+1/2}) \). A scheme for this problem is:

\[
\frac{u_{j}^{n+1} - u_{j}^{n}}{k} = \frac{1}{h^2} \left\{ \beta_{j-1/2}u_{j-1}^{n+1} + \beta_{j+1/2}u_{j+1}^{n+1} - \beta_{j-1/2}u_{j-1}^{n+1} - \beta_{j+1/2}u_{j+1}^{n+1} \right\}.
\]

Analyze the 2-norm stability of this scheme for solving this initial boundary value problem.

DO NOT NEGLECT THE FACT THAT THE PROBLEM HAS VARIABLE COEFFICIENTS AND THAT THERE ARE BOUNDARY CONDITIONS AT 0 AND 1!
Fact 1: A real symmetric $n \times n$ matrix $A$ can be diagonalized by an orthogonal similarity transformation, and $A$’s eigenvalues are real.

Fact 2: The $(N - 1) \times (N - 1)$ matrix $M$ defined by

$$
\begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 & 0 \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots & \ddots & \ddots & \ddots \\
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& & & & & & & \ddots & \ddots & \ddots & \ddots \\
& & & & & & & & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & -2 
\end{bmatrix}
$$

has eigenvalues $\mu_l = -4 \sin^2 \left( \frac{\pi l}{2N} \right)$, $l = 1, 2, \ldots, N - 1$.

Fact 3: The $(N + 1) \times (N + 1)$ matrix:

$$
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 & 0 \\
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& & & & & & & & \ddots & \ddots & \ddots \\
& & & & & & & & & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & -1 
\end{bmatrix}
$$

has eigenvalues $\mu_l = -4 \sin^2 \left( \frac{\pi l}{2(N+1)} \right)$, $l = 0, 1, \ldots, N$.

Fact 4: For a real $n \times n$ matrix $A$, the Rayleigh quotient of a vector $x \in \mathbb{R}^n$ is the scalar

$$
r(x) = \frac{x^T Ax}{x^T x}.
$$

The gradient of $r(x)$ is

$$
\nabla r(x) = \frac{2}{x^T x} (Ax - r(x)x).
$$

If $x$ is an eigenvector of $A$ then $r(x)$ is the corresponding eigenvalue and $\nabla r(x) = 0$. 
