Preliminary Exam, Numerical Analysis, August 2018

Instructions: This exam is closed book, no notes, and no electronic devices are allowed. You have three hours and you need to work on any three out of questions 1-4, and any three out of questions 5-8. All questions have equal weight and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

Problem 1. (Singular values and eigenvalues)
Let $A \in \mathbb{C}^{N\times N}$. Prove the following statements:
(a) The set of squared singular values of A equals the set of eigenvalues of A^*A.
(b) Assume that A is diagonalizable and that every eigenvalue λ of A satisfies $|\lambda| < 1$. Then:

$$\lim_{n\to\infty} \|A^n\| = 0.$$

Problem 2. (Unitary matrices)
Let $A, B \in \mathbb{C}^{N\times N}$. A and B are *unitarily equivalent* if there is a unitary matrix Q such that $A = QBQ^*$. For each of the following statements, prove that it is true, or demonstrate that it is false:
(a) If A and B are unitarily equivalent, then they have the same singular values.
(b) If A and B have the same singular values, then they are unitarily equivalent.

Problem 3. (Finite difference formulas)
Given $h > 0$, compute weights v_0 and $\{w_j\}_{j=-1}^{1}$ for the following finite difference formula for the *third* derivative $f^{(3)}(x)$:

$$\begin{aligned} f^{(3)}(x) &\approx v_0 f'(x) + \sum_{j=-1}^{1} w_j f(x+jh) \\ &= v_0 f'(x) + w_{-1} f(x-h) + w_0 f(x) + w_1 f(x+h). \end{aligned}$$

What is the order of accuracy of your formula?

Problem 4. (Lebesgue's Lemma)
Let I be a closed, bounded interval on the real line and let $\|f\|_\infty \coloneqq \sup_{x\in I} |f(x)|$. Given N distinct points $x_1, \ldots, x_N \in I$ and a continuous function f, consider the Lagrange form of the degree-$(N-1)$ polynomial interpolant of f:

$$\mathcal{I}_N f \coloneqq \sum_{j=1}^{N} f(x_j)\ell_j(x),$$

where $\{\ell_j(\cdot)\}_{j=1}^N$ are cardinal Lagrange polynomials. Show that

$$\sup_{\|g\|_\infty=1} \|\mathcal{I}_N g\|_\infty \le \Lambda := \sum_{j=1}^{N} |\ell_j(x)|,$$

and that

$$\|f - \mathcal{I}_N f\|_\infty \le (1+\Lambda) \inf_{p \in P_{N-1}} \|f - p\|_\infty,$$

where $P_{N-1} = \mathrm{span}\{1, x, \ldots, x^{N-1}\}$.

Note: In problems 5-8, the notations $k = \Delta t$ and $h = \Delta x$ are used.

Problem 5. (Elliptic Problems)

Consider the one dimensional problem for $v(x)$

$$v''(x) = f(x), \tag{1}$$

in the interval [0,1] along with homogeneous Dirichlet boundary conditions. Define the difference operator

$$\Delta_h U_j \equiv \frac{1}{h^2}\Big(U_{j-1} - 2\ U_j + U_{j+1}\Big),$$

and consider the scheme for Eq. 1

$$\Delta_h U_j = f_j$$

for $j = 1, 2, \ldots, N-1$ where $Nh = 1$, $f_j \equiv f(jh)$, and $U_0 = U_N = 0$. The approximate solution satisfies a linear system $AU = b$, where $U = (U_1, U_2, ..., U_{N-1})^T$ and $b = h^2(f_1, f_2, ..., f_{N-1})^T$.

(a) State and prove the maximum principle for any grid function $V = \{V_j\}$ with values for $j = 0, 1, ...N$, that satisfies $\Delta_h V_j \geq 0$ for $j = 1, 2..., N-1$. Sketch a grid function for which $\Delta_h V_j \geq 0$.
(b) Derive an expression for the local truncation error and find the equation that relates the local trunction error and the global error $e_j = v(x_j) - U_j$.
(c) Use the maximum principle from part (a) to show that $\|e\|_\infty = O(h^2)$ as the space step $h \to 0$.
(d) Prove that the matrix A is nonsingular.

Problem 6. (Heat Equation Stability)

Consider the variable coefficient diffusion equation

$$v_t = (\beta(x)v_x)_x, \qquad 0 < x < 1,\ t > 0$$

with Dirichlet boundary conditions

$$v(0,t) \ = \ 0, \qquad v(1,t) \ = \ 0$$

and initial data $v(x,0) = f(x)$. Assume that $\beta(x) \geq \beta_0 > 0$, and that $\beta(x)$ is smooth. Let $\beta_{j+1/2} = \beta(x_{j+1/2})$. A scheme for this problem is:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{h^2}\ \left\{\beta_{j-1/2}u_{j-1}^{n+1} - (\beta_{j-1/2} + \beta_{j+1/2})u_j^{n+1} + \beta_{j+1/2}u_{j+1}^{n+1}\right\}.$$

Analyze the 2-norm stability of this scheme for solving this initial boundary value problem.

DO NOT NEGLECT THE FACT THAT THE PROBLEM HAS VARIABLE COEFFICIENTS AND THAT THERE ARE BOUNDARY CONDITIONS AT 0 AND 1!

Problem 7. (Numerical Methods for ODE Initial Value Problems)

Consider the Linear Multistep Method

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}kf_{n+2}$$

for solving an initial value problem $y' = f(y, x)$, $y(0) = \eta$. You may assume that f is Lipschitz continuous with respect to y uniformly for all x.

(a) Analyze the consistency, stability, accuracy, and convergence properties of this method.

(b) Sketch a graph of the solution to the following initial value problem.

$$y' = -10^8[y - \cos(x)] - \sin(x), \quad y(0) = 2.$$

Would it be more reasonable to use this method or the forward Euler method for this problem? What issues should be considered in choosing a timestep k for each of the methods? Justify your answer.

Problem 8. (Higher Order Methods)

Consider the following problem for $v(x)$ on [0,1]:

$$v''(x) = f(x), \tag{2}$$

with $v(0) = v(1) = 0$. Let $N \cdot h = 1$ and define $x_j = j \cdot h$ for $j = 0, 1, ..., N$. The finite-difference scheme

$$\Delta_h U_j^h \equiv \frac{1}{h^2}\left(U_{j-1}^h - 2U_j^h + U_{j+1}^h\right) = f_j^h,$$

for $j = 1, ..., N-1$ with $U_0^h = U_N^h = 0$ gives values U_j^h that approximate $v(x_j)$ with an error of $O(h^2)$. Here the superscript h is used to indicate the grid size for the solution. Show how to use this method to find a numerical solution W_j whose values approximate $v(x_j)$ with an error of $O(h^4)$ and a numerical solution Y_j whose values approximate $v(x_j)$ with an error of $O(h^6)$.