Instructions: This exam is closed books and notes. The time allowed is three hours and you need to work on any three out of questions 1-4 and any two out of questions 5-7. All questions have equal weights and the passing score will be determined after all the exams are graded. Indicate clearly the work that you wish to be graded.

Note: In problems 5-7, the notations $k = \Delta t$ and $h = \Delta x$ are used.

1. Singular Value Decomposition (SVD):

   a) Prove the following statement:

   Singular Value Decomposition: Any matrix $A \in \mathbb{C}^{m \times n}$ can be factored as $A = U\Sigma V^*$, where $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ are unitary and $\Sigma \in \mathbb{R}^{m \times n}$ is a rectangular matrix whose only nonzero entries are non-negative entries on its diagonal.

   b) Use the SVD to prove that any matrix in $\mathbb{C}^{n \times n}$ is the limit of a sequence of matrices of full rank.

2. Linear Least Squares:

   The Linear Least Squares problem for an $m \times n$ real matrix $A$ and $b \in \mathbb{R}^m$ is the problem:

   Find $x \in \mathbb{R}^n$ such that $\|Ax - b\|_2$ is minimized.

   a) Suppose that you have data $\{(t_j, y_j)\}, j = 1, 2, \ldots, m$ that you wish to approximate by an expansion

   $$p(t) = \sum_{k=1}^{n} x_k \phi_k(t).$$

   Here, the functions $\phi_k(t)$ are given functions. Which norm on the difference between the approximation function $p$ and the data gives rise to a linear least squares problem for the unknown expansion coefficients $x_k$? What is the matrix $A$ in this case, and what is the vector $b$?

   b) Suppose that $A$ is a real $m \times n$ matrix of full rank and let $b \in \mathbb{R}^m$. What are the ‘normal equations’ for the Least Squares problem? How can they be used to solve the Least Squares problem? What is the $QR$ factorization of $A$ and how can it be used to solve the Least Squares problem? Compare and contrast these approaches for numerically solving the Least Square problem.
3. Sensitivity:

Consider a \( 6 \times 6 \) symmetric positive definite matrix \( A \) with singular values \( \sigma_1 = 1000, \sigma_2 = 500, \sigma_3 = 300, \sigma_4 = 20, \sigma_5 = 1, \sigma_6 = 0.01 \).

a) Suppose you use a Cholesky factorization package on a computer with a machine epsilon \( 10^{-14} \) to solve the system \( Ax = b \) for some nonzero vector \( b \). How many digits of accuracy do you expect in the computed solution? Justify your answer in terms of condition number and stability. You may assume that the entries of \( A \) and \( b \) are exactly represented in the computer’s floating-point system.

b) Suppose that instead you use an iterative method to find an approximate solution to \( Ax = b \) and you stop iterating and accept iterate \( x^{(k)} \) when the residual \( r^{(k)} = Ax^{(k)} - b \) has 2-norm less than \( 10^{-9} \). Give an estimate of the maximum size of the relative error in the final iterate? Justify your answer.

4. Interpolation and Integration:

a) Consider equally spaced points \( x_j = a + jh, j = 0, \ldots, n \) on the interval \([a, b]\), where \( nh = b - a \). Let \( f(x) \) be a smooth function defined on \([a, b]\). Show that there is a unique polynomial \( p(x) \) of degree \( n \) which interpolates \( f \) at all of the points \( x_j \). Derive the formula for the interpolation error at an arbitrary point \( x \) in the interval \([a, b]\):

\[
f(x) - p(x) \equiv E(x) = \frac{1}{(n + 1)!}(x - x_0)(x - x_1) \cdots (x - x_n)f^{n+1}(\eta).
\]

for some \( \eta \in [a, b] \).

b) Let \( I_n(f) \) denote the result of using the composite Trapezoidal rule to approximate \( I(f) \equiv \int_a^b f(x)dx \) using \( n \) equally sized subintervals of length \( h = (b - a)/n \). It can be shown that the integration error \( E_n(f) \equiv I(f) - I_n(f) \) satisfies

\[
E_n(f) = d_2h^2 + d_4h^4 + d_6h^6 + \ldots
\]

where \( d_2, d_4, d_6, \ldots \) are known numbers that depend only on the values of \( f \) and its derivatives at \( a \) and \( b \). Suppose you have a black-box program that, given \( f, a, b, \) and \( n \), calculates \( I_n(f) \). Show how to use this program to obtain an \( O(h^4) \) approximation and an \( O(h^6) \) approximation to \( I(f) \).
5. Elliptic Problems:

For the one dimensional Poisson problem for \( v(x) \)

\[
-v''(x) + \alpha v(x) = f(x),
\]

where \( \alpha \geq 0 \) is constant, along with Dirichlet boundary conditions in the interval \([0,1]\), consider the scheme

\[
\Delta h U_j \equiv \frac{1}{h^2} \left( -U_{j-1} + 2 U_j - U_{j+1} \right) = f_j
\]

for \( j = 1, 2, \ldots, N - 1 \) where \( Nh = 1, f_j \equiv f(jh) \), and \( U_0 = U_N = 0 \). The approximate solution satisfies a linear system \( AU = b \), where \( U = (U_1, U_2, \ldots, U_{N-1})^T \) and \( b = h^2(f_1, f_2, \ldots, f_{N-1})^T \).

a) State and prove the maximum principle for any grid function \( V = \{V_j\} \) with values for \( j = 0, 1, \ldots, N \), that satisfies \( \Delta h V_j \geq 0 \).

b) Derive the matrix \( A \) and show that it is symmetric and positive definite.

c) Use the maximum principle to show that the global error \( e_j = v(x_j) - U_j \) satisfies \( \|e\|_\infty = O(h^2) \) as the space step \( h \to 0 \).

6. Numerical Methods for ODEs:

Consider the Linear Multistep Method

\[
y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}kf_{n+2}
\]

for solving an initial value problem \( y' = f(y, x), y(0) = \eta \). You may assume that \( f \) is Lipschitz continuous with respect to \( y \) uniformly for all \( x \).

a) Analyze the consistency, stability, accuracy, and convergence properties of this method.

b) Sketch a graph of the solution to the following initial value problem.

\[
y' = -10^8[y - \cos(x)] - \sin(x), \quad y(0) = 2.
\]

Would it be more reasonable to use this method or the forward Euler method for this problem? What would you consider in choosing a timestep \( k \) for each of the methods? Justify your answer.
7. Heat Equation Stability:

a) Consider the initial value problem for the constant-coefficient diffusion equation (with \( \beta > 0 \))

\[ v_t = \beta v_{xx}, \quad t > 0 \]

with initial data \( v(x, 0) = f(x) \). A scheme for this problem is:

\[ \frac{u_j^{n+1} - u_j^n}{k} = \frac{\beta}{h^2} \left\{ u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1} \right\}. \]

Analyze the 2-norm stability of this scheme. For which values of \( k > 0 \) and \( h > 0 \) is the scheme stable? (Note that there are no boundary conditions here.)

b) Consider the variable coefficient diffusion equation

\[ v_t = (\beta(x)v_x)_x, \quad 0 < x < 1, \quad t > 0 \]

with Dirichlet boundary conditions

\[ v(0, t) = 0, \quad v(1, t) = 0 \]

and initial data \( v(x, 0) = f(x) \). Assume that \( \beta(x) \geq \beta_0 > 0 \), and that \( \beta(x) \) is smooth. Let \( \beta_{j+1/2} = \beta(x_{j+1/2}) \). A scheme for this problem is:

\[ \frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{h^2} \left\{ \beta_{j-1/2}u_{j-1}^{n+1} - (\beta_{j-1/2} + \beta_{j+1/2})u_j^{n+1} + \beta_{j+1/2}u_{j+1}^{n+1} \right\}. \]

Analyze the 2-norm stability of this scheme for solving this initial boundary value problem. DO NOT NEGLECT THE FACT THAT THERE ARE BOUNDARY CONDITIONS!
**Fact 1:** A real symmetric $n \times n$ matrix $A$ can be diagonalized by an orthogonal similarity transformation, and $A$’s eigenvalues are real.

**Fact 2:** The $(N - 1) \times (N - 1)$ matrix $M$ defined by

\[
\begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & -2 \\
\end{bmatrix}
\]

has eigenvalues $\mu_l = -4 \sin^2 \left( \frac{\pi l}{2N} \right)$, $l = 1, 2, \ldots, N - 1$.

**Fact 3:** The $(N + 1) \times (N + 1)$ matrix:

\[
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
1 & -2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & \ldots & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & -2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & -1 \\
\end{bmatrix}
\]

has eigenvalues $\mu_l = -4 \sin^2 \left( \frac{\pi l}{2(N+1)} \right)$, $l = 0, 1, \ldots, N$.

**Fact 4:** For a real $n \times n$ matrix $A$, the Rayleigh quotient of a vector $x \in \mathbb{R}^n$ is the scalar

$$r(x) = \frac{x^T A x}{x^T x}.$$  

The gradient of $r(x)$ is

$$\nabla r(x) = \frac{2}{x^T x} (A x - r(x) x).$$  

If $x$ is an eigenvector of $A$ then $r(x)$ is the corresponding eigenvalue and $\nabla r(x) = 0$.  
