Section A.

1. Let $M$ be the smooth submanifold of $\mathbb{R}^3$ consisting of points that satisfy $x^2 + y^2 - z^2 = 1$. Let $N$ be the smooth submanifold of $\mathbb{R}^3$ consisting of points that satisfy $x + y + z = 0$. Prove that $M \cap N$ is a smooth manifold.

2. Let $M$ be a smooth manifold containing a point $p$, and let $C^\infty_p(M)$ be the germs of smooth real-valued functions on $M$ at the point $p$. Prove, using only the definition of a derivation, that $\delta([\lambda]) = 0$ for any constant $\lambda \in \mathbb{R}$ and any derivation $\delta : C^\infty_p(M) \to \mathbb{R}$.

3. Let the inclusion of $\text{SO}(2)$ into $\text{SO}(3)$ be induced by the inclusion of $\mathbb{R}^2$ into $\mathbb{R}^3$. Find all connected, 2-dimensional Lie subgroups of $\text{SO}(3)$ that contain $\text{SO}(2)$.

4. Let

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Find $\exp(A)$.

5. Let $M$ and $N$ be smooth manifolds, and let $f : M \to N$ be smooth. Prove that the set of all $p \in M$ for which the differential $D_pf : T_pM \to T_{f(p)}N$ is injective is an open set.

6. Let $f : M \to N$ and $g : N \to P$ be diffeomorphisms of smooth manifolds. Let $\sigma(f)$ equal 1 if $f$ preserves orientation and $-1$ otherwise. Defining $\sigma(g)$ and $\sigma(g \circ f)$ similarly, prove that $\sigma(g \circ f) = \sigma(g)\sigma(f)$.

Section B.

7. Let $X$ be a path-connected topological space.

(a) Let $x_0, x_1$ be points in $X$, and let $\alpha : I \to X$ be any path from $x_0$ to $x_1$. Explain how $\alpha$ induces an isomorphism of groups $\alpha_* : \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(X, x_1)$. 
(b) Give an example where \(x_1 = x_0\) (so \(\alpha\) is a loop at \(x_0\)), but the map \(\alpha_*: \pi_1(X, x_0) \to \pi_1(X, x_0)\) is not the identity map.

(c) Give an example of a space \(X\) with \(H_1(X; \mathbb{Z}) = 0\) but \(\pi_1(X, x_0) \neq 0\). State carefully the results you are using.

8. Let \(n \geq 1\) be a positive integer.
   (a) Give an elementary description of the universal cover \(X \to \mathbb{R}P^n\) of \(\mathbb{R}P^n\).
   (b) Describe a CW structure on \(\mathbb{R}P^n\).
   (c) Compute the cohomology ring \(H^*(\mathbb{R}P^n; \mathbb{Z}/2)\) and the map (from (a)) \(H^*(\mathbb{R}P^n; \mathbb{Z}/2) \to H^*(X; \mathbb{Z}/2)\).

9. (a) State the van Kampen theorem for fundamental groups. Use it to compute, \(\pi_1(K, \ast)\), where \(K\) is the Klein bottle, and \(\pi_1(S_g, \ast)\), where \(S_g\) is the orientable surface of genus \(g\). (Here \(\ast\) is any base-point.)
   (b) Show that \(S_g\) is indeed orientable; carefully state the results you are using.

10. Let \(C_\bullet\) be a chain complex of finitely-generated \(\mathbb{Z}\)-modules such that \(C_i = 0\) for all \(|i| \gg 0\) (i.e., the complex \(C_\bullet\) is bounded). Define the Euler characteristic of \(C\) to be

\[
\chi(C) = \sum_{i \in \mathbb{Z}} (-1)^i \text{rk}(C_i),
\]

where \(\text{rk}(C_i)\) denotes the rank of the (finitely-generated) abelian group \(C_i\).

   (a) Let \(X\) be a finite CW complex, i.e. one having finitely many cells in total. Define \(\chi(X)\) to be \(\chi(C^{\text{CW}}_\bullet(X))\), where \(C^{\text{CW}}_\bullet(X)\) denotes the cellular chain complex of \(X\). Show that

\[
\chi(X) = \sum_{i \in \mathbb{Z}} (-1)^i \text{rk}(H_i(X)),
\]

and that this quantity equals \(\sum_{i \in \mathbb{Z}} (-1)^i \dim_F(H_i(X; F))\) for any field \(F\).

   (b) Let \(X\) be a connected compact manifold of odd dimension; you may assume \(X\) is homeomorphic to a finite CW complex. Show that \(\chi(X) = 0\).