DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

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Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. To pass the exam you need to pass **both** parts.

A. Answer all of the following questions.

1. Let $A : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 4 \\ 0 & -1 \end{pmatrix}$$

Find λ such that $A^*(3dy_1 \wedge dy_2 - 2dy_2 \wedge dy_3 + 7dy_1 \wedge dy_3) = \lambda dx_1 \wedge dx_2$.

2. Let Σ_g be a closed, orientable surface of genus g, and let $T\Sigma_g$ be its tangent bundle. Prove that $T\Sigma_g$ is trivial if and only if $g = 1$.

3. Let M be a compact manifold with a vector field $X \in \Gamma(TM)$ and resulting one-parameter group $\{\theta_t^X\}_{t\in\mathbb{R}} \leq \mathrm{Diff}(M)$. For any $f \in \mathrm{Diff}(M)$, let $f_*X \in \Gamma(TM)$ be the pushforward of X. Show that the one-parameter group for f_*X is $\{f \circ \theta_t^X \circ f^{-1}\}_{t\in\mathbb{R}}$.

4. Specify the leaves of a 2-dimensional foliation on $S^2 \times S^1$, such that not every leaf is compact.

5. Find all connected subgroups of $\mathrm{SL}_2(\mathbb{R})$ that contain the group $V = \{\, g \in \mathrm{SL}_2(\mathbb{R}) \mid g_{1,2} = 0 \text{ and } g_{1,1} = g_{2,2} = 1 \,\}$.

6. Let G be a Lie group. Prove that G is orientable, and that if $L_g \in \mathrm{Diff}(G)$ is left multiplication by $g \in G$, then L_g preserves orientation.

B. Answer all of the following questions.

7. Let Σ_g be a closed, orientable surface of genus g. Given $p \geq 1$ distinct points $\{x_1, \ldots x_p\} \subseteq \Sigma_g$, let $\Sigma_{g,p} = \Sigma_g - \{x_1, \ldots x_p\}$. Find the cohomology of $\Sigma_{g,p}$ with integer coefficients, including its cup product structure.

8. For $k \in \mathbb{N}$ and a topological space X, let C_kX be a collection of k cones on X, with their bases – each homeomorphic to X – identified. So for example, C_2X is the suspension of X. Express the homology groups (with integer coefficients) of C_kX in terms of the homology groups of X.

9. Use covering spaces to write the generators of a normal rank 3 free subgroup of a rank 2 free group. Use covering spaces to write the generators of a non-normal rank 3 free subgroup of a rank 2 free group.

10. Let $n, k \in \mathbb{N}$ with $n \geq 2$. Suppose $f : \mathbb{R}P^n \to T^k$ is continuous. Determine $f_* : H_m(\mathbb{R}P^n) \to H_m(T^k)$ for all m.

11. Give the chain complex used to define cellular homology of a cell complex X. Prove that it's a chain complex.

12. Let D^n be a closed n-dimensional disk. Prove that if f is a homeomorphism of D^n, then f fixes some $x \in D^n$.