UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Geometry / Topology

Jan. 4, 2019.

Instructions This exam has two parts, A and B, covering material from Math 6510 and 6520, respectively. To pass the exam, you will have to pass each part. To pass each part you will have to demonstrate mastery of the material. It will be up to the faculty grading the exam to determine if sufficient understanding of the material is demonstrated. Please answer as many questions as you can. Not all questions are equally difficult. Each question is worth 20 points. A passing score is 60 on each part.

Part A. Do all problems.

1. Let M and N be smooth manifolds and let M be compact. Let $F : M \times \mathbb{R} \to N$ be a smooth function and for any $t \in \mathbb{R}$, let $F_t : M \to N$ be the function defined by $F_t(x) = F(x,t)$. If F_0 is a submersion, prove that there is some $\epsilon > 0$ such that F_t is an submersion for all $t \in (-\epsilon, \epsilon)$.

2. Find all connected Lie subgroups of $\mathbb{SL}_2(\mathbb{R})$ that contains the subgroup
$$H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}.$$

3. Let $\alpha = y\,dx - x\,dy + dz$ be a one form on $\mathbb{R}^3$. Define a 2-plane field distribution at $p \in \mathbb{R}^3$ by $\Delta_p = \{V \in T_p\mathbb{R}^3 : \alpha(V) = 0\}$. Is the plane field Δ_p integrable? Why?

4. Let $a : \mathbb{S}^2 \to \mathbb{S}^2$ be the antipodal map $a(x) = -x$ and let
$$p : \mathbb{S}^2 \to \mathbb{RP}^2 = \mathbb{S}^2/x \sim -x, \qquad p(x) = [x]$$
be the projection to the projective plane. Let ω be a two form on $\mathbb{RP}^2$.
   (a) Prove that $\int_{\mathbb{S}^2} p^*\omega = 0$. [Hint: $p^*\omega$ is a-invariant.]
   (b) Prove that there is a one form η on $\mathbb{RP}^2$ such that $d\eta = \omega$. You are allowed to use the fact that if β is a two-form on $\mathbb{S}^2$ and $\int_{\mathbb{S}^2} \beta = 0$ then β is exact, but you are **not** allowed to use the de Rham Theorem.
   (c) Prove that $\mathbb{RP}^2$ is not orientable.

5. Let M be a compact, smooth two dimensional manifold without boundary embedded in $\mathbb{R}^3$. Show that for almost every two-plane $L \in \mathbb{R}^3$ such that $L \cap M$is nonempty, the intersection $L \cap M$ consists of finitely many closed curves.

Part B. Do all problems.

1. Let $X = S^1 \vee S^1$, with basepoint $*$ at the point where the two circles are glued, and with $a, b\colon S^1 \to X$ the generators of $\pi_1(X, *)$ corresponding to the identity maps onto each circle in the wedge sum. Construct a covering space $p\colon (E, e) \to (X, *)$ such that $p_*\pi_1(E, e)$ is the subgroup of $\pi_1(X, *)$ generated by a^2, ab^{-1}, ab. Justify that your answer works *in two ways*:

   (a) using covering space theory; and

   (b) using the van Kampen theorem.

2. Let G be any group. Construct a connected topological space whose fundamental group is isomorphic to G. Justify your answer.

3. (a) Explain how the set of lines in $\mathbb{C}^{n+1}$ is topologized to yield the space $\mathbb{CP}^n$. Describe how to equip $\mathbb{CP}^n$ with a CW structure with m-skeleton isomorphic to $\mathbb{CP}^m$ for all $m \leq n$.

   (b) Compute the singular homology groups $H_*(\mathbb{CP}^n; \mathbb{Z})$. Be sure you indicate which properties of homology you are using.

4. Fix integers $n \geq 1$ and $m \geq 1$.

   (a) Construct a connected space X such that

   $$\widetilde{H}_r(X; \mathbb{Z}) = \begin{cases} 0 \text{ if } r \neq n. \\ \mathbb{Z}/m \text{ if } r = n. \end{cases}$$

   (Here $\widetilde{H}$ denotes reduced homology.)

   (b) For any connected space X satisfying the conclusion of part (a), compute the groups $H_r(X; \mathbb{Z}/2)$ for all $r \geq 0$.

5. Let X be a connected compact orientable manifold of dimension n. Explain carefully how Poincaré duality implies the existence of a bilinear pairing

   $$H^k(X; \mathbb{Z}) \times H^{n-k}(X; \mathbb{Z}) \to \mathbb{Z}$$

   that modulo torsion becomes a perfect (i.e., non-singular) pairing. Deduce from this the structure of the cohomology *ring* $H^*(\mathbb{CP}^n; \mathbb{Z})$.