DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

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Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. To pass the exam you need to pass **both** parts.

A. Answer all of the following questions.

1. Let B be the open ball in $\mathbb{R}^n$ that is centered at 0 and has radius 1. If $y \in B$ and $||y|| < 1/2$, define a vector field on B and use the resulting flow to find a diffeomorphism $h : B \to B$ with $h(0) = y$ such that h is homotopic to the identity map on B. Conclude that if N is a smooth connected manifold and $p, q \in N$, then there is a diffeomorphism $H : N \to N$ such that $H(p) = q$ and such that H is homotopic to the identity map on N.

2. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be linear and let $\phi \in \Lambda^n(\mathbb{R}^n)$. Prove $A^*\phi = \det(A)\phi$.

3. Let $\Phi : \mathfrak{sl}_2(\mathbb{R}) \to \mathbb{R}$ be a Lie algebra homomorphism, where $\mathfrak{sl}_2(\mathbb{R})$ is the Lie algebra of $\mathrm{SL}_2(\mathbb{R})$ and $\mathbb{R}$ is the Lie algebra of the Lie group $\mathbb{R}$. Prove that $\Phi = 0$.

4. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ such that $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Prove that G is abelian.

5. Prove that S^n is a smooth manifold for any n.

6. Let M and N be smooth, compact, oriented manifolds without boundary, both of dimension n. Suppose that $f : M \to N$ and $g : M \to N$ are smoothly homotopic. Prove that for any $\omega \in \Omega^n(N)$, we have $\int_M f^*\omega = \int_M g^*\omega$.

B. Answer all of the following questions.

7. Prove that the fundamental group of a finite connected graph is a free group.

8. Show that the union of circles in $\mathbb{R}^2$ with centers at $(0, \frac{1}{2n})$ and radius $\frac{1}{n}$ does not have a universal cover.

9. Show that S^n has a continuous field of nonzero tangent vectors iff $n > 0$ is odd. Conclude that the even-dimensional spheres are not Lie groups.

10. Construct a Δ-complex structure on $\mathbb{RP}^3$ and use it to compute the cohomology rings of $\mathbb{RP}^3$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients.

11. Prove that nonempty open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic if $m \neq n$.

12. Let M be a closed, orientable 3-manifold such that
$$H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_3$$
Calculate the remaining homology and cohomology groups for M with $\mathbb{Z}$ coefficients.