DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
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Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. To pass the exam you need to pass **both** parts.

A. Answer all of the following questions.

1. Let M be a smooth compact manifold without boundary and $\Delta \subset M \times M$ the diagonal. Show that Δ is not the boundary of a compact manifold $W \subset M \times M$.

2. Define $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $\omega(x,y) = x_1y_1 + \cdots + x_{n-1}y_{n-1} - x_ny_n$ and define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \omega(x,x)$. Show that $M = f^{-1}(-1)$ is a smooth manifold. Viewing ω as a symmetric 2-tensor on $\mathbb{R}^n$ show that the restriction of ω to M is positive definite. (Hint: First assume $y \in T_xM$ and that $x_n + y_n = 0$ and show that $\omega(x+y, x+y) = -1 + \omega(y,y) \geq 0$.)

3. Let $S^2 = \{(x,y,z) : x^2 + y^2 + z^2 = 1\}$ be the standard 2-sphere in $\mathbb{R}^3$ and $f : S^2 \to \mathbb{R}$ the restriction of the function $(x,y,z) \mapsto x^2 + 2y^2 + 3z^2$ to S^2. Find the critical points of f and show that they are non-degenerate. (Recall that a critical point is non-degenerate if in some coordinate chart the determinant of the Hessian is non-zero.)

4. Let W be a vector field on a smooth manifold M and assume that V has a flow that is defined on all of M and for all time. Let V be another vector field on M such that $V - W$ has compact support. Show that V has a flow on all of M defined for all time.

5. Let $S^1 = \{(x,y) : x^2 + y^2 = 1\}$ and define $f : \mathbb{R} \to S^1$ by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. Show there exists a unique 1-form ω on S^1 such that $f^*\omega = dt$. Show that ω is closed but not exact.

6. Let M be a smooth manifold. Show that the tangent bundle TM is orientable.

B. Answer all of the following questions.

7. Construct an explicit irregular (i.e. not normal) covering space of the Klein bottle.

8. Let X be the space obtained from the disjoint union of a circle S^1 and a cylinder $S^1 \times [0,1]$ by attaching the cylinder along its boundary to the circle with the attaching map on $S^1 \times \{0\}$ having degree 2 and on $S^1 \times \{1\}$ degree 3. Find a presentation of the fundamental group of X.

9. Let Y be a space obtained from a space X by attaching an n-cell. What is the relationship between the homology groups of X and of Y? Specifically, prove the following, where $f : X \to Y$ is inclusion.
   - $f_* : H_i(X) \to H_i(Y)$ is an isomorphism for $i \neq n-1, n$.
   - $f_* : H_{n-1}(X) \to H_{n-1}(Y)$ is surjective.

- $f_* : H_n(X) \to H_n(Y)$ is injective.
- Either $f_* : H_{n-1}(X) \to H_{n-1}(Y)$ has finite kernel or $f_* : H_n(X) \to H_n(Y)$ is an isomorphism (but not both).

Assuming the ranks of $H_{n-1}(X)$ and $H_n(X)$ are finite, also prove that

$$\text{rank }(H_n(X)) - \text{rank }(H_{n-1}(X)) = \text{rank }(H_n(Y)) - \text{rank }(H_{n-1}(Y)) - 1$$

10. Let $p : \tilde{X} \to X$ be an n-sheeted covering map between two connected, locally path-connected spaces. Let $C_*(X)$ and $C_*(\tilde{X})$ denote the singular chain complexes of X and $\tilde{X}$ respectively.
    (a) Give a definition of a chain morphism and prove that $\tau : C_*(X) \to C_*(\tilde{X})$ given by

    $$\tau(\sigma) = \sum_{i=1}^{n} \hat{\sigma}_i$$

    is a chain morphism, where $\hat{\sigma}_1, \cdots, \hat{\sigma}_n$ are the n lifts of σ to $\tilde{X}$, and σ is a singular simplex in X.
    (b) Prove that

    $$p_* : H_i(\tilde{X}; \mathbb{Q}) \to H_i(X; \mathbb{Q})$$

    is always surjective. Hint: Consider the composition $p_*\tau : C_*(X) \to C_*(X)$.
    (c) Find an example where

    $$p_* : H_i(\tilde{X}; \mathbb{Z}) \to H_i(X; \mathbb{Z})$$

    is not surjective.

11. Prove that there is no map $h : \mathbb{R}P^3 \to \mathbb{R}P^2$ that induces an isomorphism between fundamental groups. Hint: use cup products with $\mathbb{Z}/2\mathbb{Z}$ coefficients. State all theorems you are using.

12. Let M be a connected compact 3-manifold (without boundary) and assume that $H_1(M) = \mathbb{Z}/3\mathbb{Z}$. Find $H_2(M)$. Carefully justify the answer, and state all theorems you are using.