DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

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Instructions: Do all problems from section A and six (6) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited.

A. Answer all of the following questions.

1. (a) Let M be a smooth manifold. What's the definition of an orientation on M?

(b) Suppose M is a smooth oriented manifold with boundary. What is the induced orientation on ∂M?

(c) State Stokes' Theorem.

(d) Let M be a smooth manifold without boundary that is oriented by the form $\omega \in \Omega^{\dim(M)}(M)$. Suppose that $[0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ is oriented by the form dx, and that $M \times [0,1]$ has the orientation defined by $\omega \wedge dx$. Endow $\partial(M \times [0,1]) = (M \times \{0\}) \cup (M \times \{1\})$ with the orientation induced by $M \times [0,1]$ and let $i_0 : M \to M \times \{0\}$ and $i_1 : M \to M \times \{1\}$ be the obvious diffeomorphisms. Prove that i_0 preserves orientation if and only if i_1 does not.

(e) Let $f : M \to N$ and $g : M \to N$ be smooth maps between smooth manifolds without boundaries. What is the definition of f and g being smoothly homotopic to each other?

(f) Suppose M is a smooth, compact, oriented manifold without boundary. Suppose N is a smooth manifold and that $f : M \to N$ and $g : M \to N$ are smoothly homotopic smooth maps. Prove that if $\theta \in \Omega^k(N)$ for some $k > \dim(M)$, then

$$\int_M f^*\theta = \int_M g^*\theta$$

2. (a) What is a subalgebra of a Lie algebra?

(b) Let G be a Lie group, with Lie algebra $\mathfrak{g}$. Prove that any Lie subgroup H of G corresponds to a subalgebra of $\mathfrak{g}$.

(c) What is a smooth plane field on G?

(d) Give an example of a foliation on a 2-torus that has compact leaves and noncompact leaves.

(e) State Frobenius' theorem.

(f) Prove that any subalgebra of $\mathfrak{g}$ corresponds to a foliation of G.

(g) Given a subalgebra of $\mathfrak{g}$, and the corresponding foliation of G, prove that the leaf containing the identity is a group, and thus that there is a correspondence between connected Lie subgroups of G and subalgebras of $\mathfrak{g}$.

B. Answer six of the following questions.

3. Define a deformation retract. Construct a 2-dimensional cell complex that contains both an annulus $S^1 \times I$ and a Möbius band as deformation retracts.

4. Let G be a Lie group and $1 \in G$ the identity element. Prove that $\pi_1(G, 1)$ is an abelian group.

5. Let $f : S^1 \to S^1$ be given by $f(z) = z^2$, where we regard S^1 as the unit circle in $\mathbb{C}$. Find a presentation of the fundamental group of the mapping torus
$$M_f = S^1 \times I/(z, 1) \sim (f(z), 0)$$
of f.

6. Suppose that X and Y are two connected manifolds or cell complexes homotopy equivalent to each other. Prove that their universal covers $\tilde{X}$ and $\tilde{Y}$ are homotopy equivalent to each other. Carefully state all theorems you are using.

7. An algebraic fact, often used in homology theory, is that a short exact sequence of chain complexes induces a long exact sequence in homology. State this fact carefully and give a definition of the "connecting homomorphism" (the one that lowers the degree of the homology group).

8. Let X denote the solid torus $S^1 \times D^2$. Using Mayer-Vietoris, compute the homology groups of the double of X, obtained by taking two copies of X and gluing them along the boundary via the identity map. Also compute $H_i(X, \partial X)$.

9. Let X be a finite connected cell complex such that $H^2(X; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $H^i(X; \mathbb{Z}) = 0$ for $i = 1$ and $i > 2$. Compute $H_i(X; \mathbb{Z})$ for all i. State all theorems you are using.

10. Let M be a closed orientable connected 4-manifold with $H^1(M) = H^3(M) = 0$ and $H^2(M) = H^4(M) = \mathbb{Z}$. What are the possible cup product structures on $H^*(M)$? State all theorems you are using. All cohomology groups are with integral coefficients.