DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

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Instructions: Do all problems from section A and four (4) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited.

A. Answer all of the following questions.

1. What is the definition of a foliation on a manifold? Give an example of a foliation on a manifold.

2. Find the Lie bracket $[\,2\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\,,\, x\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial z}\,]$.
   Is the plane field $\Delta_{(x,y,z)} = \text{span}\{\,2\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}\,,\, x\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial z}\,\}$ integrable?

3. Find $a, b \in C^\infty(\mathbb{R}^3)$ such that
$$d(xyzdx \wedge dy + 3x^3 dy \wedge dz) = adx \wedge dy \wedge dz$$
   and
$$dx \wedge (14xdz) \wedge dy = bdx \wedge dy \wedge dz$$

4. Let $f : T^2 \to \mathbb{R}^3$ be an embedding. Find $\int_{T^2} dx \wedge dy$.

5. Let G be a Lie group with a discrete subgroup $\Gamma \leq G$. Prove that Γ acts freely and properly discontinuously on G. (It will follow that $\Gamma\backslash G$ is a smooth manifold, but you don't have to show this.)

6. Let $f : S^2 \to S^2$ be smooth and surjective. Prove that there is an open set $U \subseteq S^2$ such that $f : U \to f(U)$ is a diffeomorphism.

7. Draw a nowhere vanishing vector field on S^1. Explain why there is a nowhere vanishing vector field on S^3.

8. Find the Lie derivative $L_{y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}}(xy)$.

9. Find
$$\exp\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
   and
$$\exp\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

10. Let $G \cong \mathrm{SO}(2) \ltimes \mathbb{R}^4$ be the Lie group given by matrices

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) & x & y \\ \sin(\theta) & \cos(\theta) & z & w \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}$$

where $\theta, x, y, z, w \in \mathbb{R}$. Find the linear subspace of $\mathfrak{gl}_4(\mathbb{R})$ that is the Lie algebra of G.

11. What is the Lie algebra of $\mathrm{SO}(n)$?

B. Answer four of the following questions.

12. Let X be a path connected, locally path connected, semi-locally simply connected Hausdorff space. One of the main theorems of covering space theory asserts that there is a simply connected space $\tilde{X}$ and a covering map $p : \tilde{X} \to X$. Outline the construction of $\tilde{X}$ and p as follows:
    (a) Describe $\tilde{X}$ as a set.
    (b) Describe the topology on $\tilde{X}$.
    (c) Describe p. You do not need to prove that p is a covering map.
    (d) Outline the proof that $\tilde{X}$ is simply connected.

13. Let $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and $S^1 \subset S^3$ is given by $w = 0$. Let X be the quotient space of S^3 obtained by collapsing S^1 to a point. Use any method to compute $H_k(X; \mathbb{Z})$ for all $k \in \mathbb{Z}$.

14. Define the concepts of chain maps and chain homotopies, and prove that chain homotopy is an equivalence relation.

15. Let G be a group of homeomorphisms acting freely on the n-sphere S^n with n even. Prove that G has at most two elements.

16. Give an example of a connected cell complex X such that $H_k(X; \mathbb{Z}) = 0$ for all $k > 0$ but $\pi_1(X) \neq \{1\}$. You do not have to verify the second claim, but you should argue the first.

17. Prove that a closed orientable surface S_g of genus $g \geq 1$ is not homotopy equivalent to the wedge $X \vee Y$ of two finite cell complexes both of which have nontrivial $H_1(\cdot; \mathbb{Z})$.

18. Let M be a connected topological n-manifold. Define what it means for M to be orientable. Then state the Poincaré duality theorem (including a careful definition of the map that is claimed to be an isomorphism).

19. Let X be a cell complex obtained from the circle by attaching a 2-cell with a degree 3 attaching map. Prove that X cannot be embedded into $\mathbb{R}^3$. You may assume that the embedding is such that there exists a "regular neighborhood", i.e. a smooth compact manifold $N \subset \mathbb{R}^3$ such that $X \hookrightarrow N$ is a homotopy equivalence.