A. Answer all of the following questions.

1. What is the definition of a foliation on a manifold? Give an example of a foliation on a manifold.

2. Find the Lie bracket \[ [ \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} , x \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} ] \].
   Is the plane field \( \Delta_{(x,y,z)} = \text{span}\{2 \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}, x \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial z}\} \) integrable?

3. Find \( a, b \in C^\infty(\mathbb{R}^3) \) such that
   \[ d(xyz dx \wedge dy + 3x^3 dy \wedge dz) = adx \wedge dy \wedge dz \]
   and
   \[ dx \wedge (14xdz) \wedge dy = bdx \wedge dy \wedge dz \]

4. Let \( f : T^2 \rightarrow \mathbb{R}^3 \) be an embedding. Find \( \int_{T^2} dx \wedge dy \).

5. Let \( G \) be a Lie group with a discrete subgroup \( \Gamma \leq G \). Prove that \( \Gamma \) acts freely and properly discontinuously on \( G \). (It will follow that \( \Gamma \backslash G \) is a smooth manifold, but you don’t have to show this.)

6. Let \( f : S^2 \rightarrow S^2 \) be smooth and surjective. Prove that there is an open set \( U \subseteq S^2 \) such that \( f : U \rightarrow f(U) \) is a diffeomorphism.

7. Draw a nowhere vanishing vector field on \( S^1 \). Explain why there is a nowhere vanishing vector field on \( S^3 \).

8. Find the Lie derivative \( L_{y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}} (xy) \).

9. Find
   \[ \exp \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]
   and
   \[ \exp \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{pmatrix} \]
10. Let $G \cong \text{SO}(2) \ltimes \mathbb{R}^4$ be the Lie group given by matrices
\[
\begin{pmatrix}
\cos(\theta) & -\sin(\theta) & x & y \\
\sin(\theta) & \cos(\theta) & z & w \\
0 & 0 & \cos(\theta) & -\sin(\theta) \\
0 & 0 & \sin(\theta) & \cos(\theta)
\end{pmatrix}
\]
where $\theta, x, y, z, w \in \mathbb{R}$. Find the linear subspace of $\mathfrak{gl}_4(\mathbb{R})$ that is the Lie algebra of $G$.

11. What is the Lie algebra of $\text{SO}(n)$?

B. Answer four of the following questions.

12. Let $X$ be a path connected, locally path connected, semi-locally simply connected Hausdorff space. One of the main theorems of covering space theory asserts that there is a simply connected space $\tilde{X}$ and a covering map $p : \tilde{X} \to X$. Outline the construction of $\tilde{X}$ and $p$ as follows:
   (a) Describe $\tilde{X}$ as a set.
   (b) Describe the topology on $\tilde{X}$.
   (c) Describe $p$. You do not need to prove that $p$ is a covering map.
   (d) Outline the proof that $\tilde{X}$ is simply connected.

13. Let $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and $S^1 \subset S^3$ is given by $w = 0$. Let $X$ be the quotient space of $S^3$ obtained by collapsing $S^1$ to a point. Use any method to compute $H_k(X; \mathbb{Z})$ for all $k \in \mathbb{Z}$.

14. Define the concepts of chain maps and chain homotopies, and prove that chain homotopy is an equivalence relation.

15. Let $G$ be a group of homeomorphisms acting freely on the $n$-sphere $S^n$ with $n$ even. Prove that $G$ has at most two elements.

16. Give an example of a connected cell complex $X$ such that $H_k(X; \mathbb{Z}) = 0$ for all $k > 0$ but $\pi_1(X) \neq \{1\}$. You do not have to verify the second claim, but you should argue the first.

17. Prove that a closed orientable surface $S_g$ of genus $g \geq 1$ is not homotopy equivalent to the wedge $X \vee Y$ of two finite cell complexes both of which have nontrivial $H_1(\cdot; \mathbb{Z})$.

18. Let $M$ be a connected topological $n$-manifold. Define what it means for $M$ to be orientable. Then state the Poincaré duality theorem (including a careful definition of the map that is claimed to be an isomorphism).

19. Let $X$ be a cell complex obtained from the circle by attaching a 2-cell with a degree 3 attaching map. Prove that $X$ cannot be embedded into $\mathbb{R}^3$. You may assume that the embedding is such that there exists a “regular neighborhood”, i.e. a smooth compact manifold $N \subset \mathbb{R}^3$ such that $X \hookrightarrow N$ is a homotopy equivalence.