Instructions: Provide solutions for as many problems as you can in the time allowed. Divide your efforts on both sections, A and B, as you'll have to pass both parts to pass the qualifying exam as a whole. Cite the theorems that you use.

Section A.
1. Let $\mathbb{P}^n(\mathbb{R})$ be $n$-dimensional real projective space. Use an explicit collection of charts to prove that $\mathbb{P}^n(\mathbb{R})$ is a smooth manifold.

2. Prove that there is no immersion of the $n$-sphere into $\mathbb{R}^n$.

3. Let $M$ be the smooth submanifold of $\mathbb{R}^3$ that is the solution set of the equation $x^2 + y^2 - z^2 = 1$. Let $N$ be the smooth submanifold of $\mathbb{R}^3$ that is the solution set of the equation $x^2 + y^2 + z^2 = 1$. Prove that $M$ and $N$ do not intersect transversally in $\mathbb{R}^3$.

4. Let $M$ and $N$ be smooth manifolds, and let $M$ be compact. Suppose $F : M \times [0, 1] \to N$ is a smooth function, and for any $t \in [0, 1]$ let $F_t : M \to N$ be the function defined by $F_t(p) = F(p, t)$. If $F_0$ is an immersion, prove that there is some $\varepsilon \in (0, 1]$ such that $F_t$ is an immersion for all $t \in [0, \varepsilon)$.

5. Let $G$ be a Lie group, and let $H$ be the connected component of $G$ that contains the identity. Prove that $H$ is a Lie subgroup of $G$, and that $H$ is a normal subgroup of $G$. (Do not use the theorem that closed subgroups of Lie groups are Lie groups.)

6. Let $\Sigma$ be a closed, smooth surface. Show that if $\Sigma$ admits a nonvanishing vector field, then the Euler characteristic of $\Sigma$ equals 0.

Section B.
7. Prove that if $X$ and $Y$ are path connected spaces, then $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

8. Let $\Sigma_g$ be a closed, orientable surface of genus $g \geq 1$. Prove that $\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid \prod_{i=1}^{2g} [a_i, b_i] \rangle$.

9. Show that $\pi_1(\Sigma_2)$ contains $\pi_1(\Sigma_3)$ as a normal subgroup.

10. Let $D^k$ be the closed $k$-dimensional disk. Use that $D^k/\partial D^k = S^k$ to find $H_n(S^k; \mathbb{Z})$ for all $n$.

11. Suppose $f : S^k \to S^k$ is continuous and not surjective. Prove that $f_* : H_k(S^k; \mathbb{Z}) \to H_k(S^k; \mathbb{Z})$ is the zero homomorphism.

12. Suppose $M$ is a closed, orientable, simply connected, smooth manifold of dimension 3. Let $S_3$ be the symmetric group on 3 letters, and suppose $S_3$ acts on $M$ freely, by orientation preserving diffeomorphisms. Let $N = S_3 \setminus M$, and find $H_n(N; \mathbb{Z})$ and $H^n(N; \mathbb{Z})$ for all $n$. 