Instructions: Do all problems from section A. Be sure to provide all relevant definitions and statements of theorems cited. To pass the exam you need to have at least 3 completely correct solutions in part A along with passing part B. If you don’t pass B but get 4 problems from part A correct you will have passed that section of the exam.

A. Answer all of the following questions.

1. (a) State the definition of a regular value and state the pre-image theorem.
(b) Let \( F : \mathbb{R}^n \to \mathbb{R}^m \) be a smooth function with \( y \in \mathbb{R}^m \) a regular value and let \( M = F^{-1}(y) \). If \( f : \mathbb{R}^n \to \mathbb{R}^k \) is a smooth function and \( x \in M \) is critical point for \( f|_M \) what are the possible values for the dimension of \( \ker F^*(x) \cap \ker f^*(x) \)? (Here \( F^*(x) \) is the tangent map from \( T_x \mathbb{R}^n \to T_y \mathbb{R}^m \) and \( f^*(x) \) is also a tangent map.)
(c) Define \( G : \mathbb{R}^n \to \mathbb{R}^{m+k} \) by \( G(x) = (F(x), f(x)) \).
   (i) If \( z \in \mathbb{R}^k \) is regular value of \( f|_M \) is \( (y, z) \in \mathbb{R}^{m+k} \) always a regular value of \( G \)?
   Give a proof or find a counterexample.
   (ii) If \( z \in \mathbb{R}^k \) is a regular of \( f \) (as a function on \( \mathbb{R}^n \)) is \( (y, z) \in \mathbb{R}^{m+k} \) always a regular value of \( G \). Give a proof or find a counterexample.

2. Let \( M \) be a differentiable manifold. Prove that its tangent bundle \( TM \) and its cotangent bundle \( TM^* \) are isomorphic as smooth vector bundles.

3. Let \( V \) be a smooth vector field on \( \mathbb{R}^2 \) and assume that outside of a compact set \( V = \frac{\partial}{\partial x} \). Show that the flow for \( V \) is defined for all time.

4. (a) State Stokes theorem.
(b) If \( \omega \in \Omega^n(\mathbb{R}^n) \) has compact support and \( \int_{\mathbb{R}^n} \omega \neq 0 \) show that there does not exist an \( \alpha \in \Omega^{n-1}(\mathbb{R}^n) \) with compact support and \( d\alpha = \omega \).
(c) Now assume that \( n = 1 \) and that \( \int_{\mathbb{R}} \omega = 0 \). Find an \( \alpha \in \Omega^0(\mathbb{R}) \) with compact support and \( d\alpha = \omega \).

5. Let \( S^2 = \{(x, y, z) \in \mathbb{R}^3| x^2 + y^2 + z^2 \} \). Define \( \pi : \mathbb{R}^3 \setminus \{0\} \to S^2 \) by \( \pi(x, y, z) = \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) \) and \( T_\epsilon(x, y, z) = (x, y, z + \epsilon) \). For \( \epsilon \in (0, 1) \) the map \( f_\epsilon = \pi \circ T_\epsilon \) is a Lefschetz map from \( S^2 \) to itself. Calculate its Lefschetz number and conclude every map of \( S^2 \) to itself that is homotopic to the identity has a fixed point. (Hint: It will be easier to calculate the derivative of \( \pi \) and \( T_\epsilon \) separately and use the chain rule to find the derivative of \( f_\epsilon \).

6. Let \( \omega \) and \( \eta \) be closed forms on a manifold \( M \). Show that the de Rham cohomology class of \( \omega \wedge \eta \) only depends on the cohomology classes of \( \omega \) and \( \eta \).