A. Answer all of the following questions.

1. Let $S^1$ denote the unit circle in $\mathbb{C}$ and define $f : S^1 \times S^1 \rightarrow \mathbb{R}$ by $f(z, w) = \text{Re}(z) + \text{Re}(w)$. Show that $f$ is a Morse function and compute the indices of the critical points.

2. Give an example of a 1-form on $\mathbb{R}^2 - \{(0, 0), (1, 0)\}$ which is closed but not exact and prove both properties.

3. Let $S \subset \mathbb{R}^3$ be a (nonempty) smooth surface. Prove that there exists a plane $ax + by + cz = d$ in $\mathbb{R}^3$ whose intersection with $S$ is a collection of circles and lines.

4. Let $\Delta$ be a plane field on a manifold $M$ and let $X, Y$ be two vector fields tangent to $\Delta$. Show that if for some $p \in M$ we have $Y_p = 0$ then $[X, Y]_p \in \Delta$.

5. Prove that every manifold admits a Riemannian metric. Use partitions of unity and not embedding into $\mathbb{R}^N$.

6. Regard $SO(n)$ as a subset of the space of $n \times n$ matrices, which in turn is identified with $\mathbb{R}^{n^2}$. Show that $SO(n)$ is a submanifold and identify the tangent space to $SO(n)$ at the identity $I$ (i.e. the Lie algebra of $SO(n)$).
B. Answer all of the following questions.

7. Let $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and $S^1 \subset S^3$ is given by $w = 0$. Let $X$ be the quotient space of $S^3$ obtained by collapsing $S^1$ to a point. Use any method to compute $H_k(X; \mathbb{Z})$ for all $k \in \mathbb{Z}$.

8. Let $S_g$ denote the closed oriented surface of genus $g$. Show that if $h < g$ then every map $S_h \to S_g$ has degree 0. Does every such map have to be null-homotopic?

9. Define the concepts of chain maps and chain homotopies, and prove that chain homotopy is an equivalence relation.

10. Let $X$ be a path connected, locally path connected, semi-locally simply connected Hausdorff space. One of the main theorems of covering space theory asserts that there is a simply connected space $\tilde{X}$ and a covering map $p : \tilde{X} \to X$. Outline the construction of $\tilde{X}$ and $p$ as follows:
   (a) Describe $\tilde{X}$ as a set.
   (b) Describe the topology on $\tilde{X}$.
   (c) Describe $p$. You do not need to prove that $p$ is a covering map.
   (d) Outline the proof that $\tilde{X}$ is simply connected.

11. Let $p : \tilde{X} \to X$ be a covering map between cell complexes. Assume that $\tilde{X}$ is connected and that $p$ is null-homotopic. Prove that $\tilde{X}$ is contractible.

12. Give an example of a space $X$ such that $H_i(X; \mathbb{Z}/2) = 0$ for $i > 0$ but for some $i > 0$ $H_i(X; \mathbb{Z}) \neq 0$. Is it possible to have a space $X$ with $H_i(X; \mathbb{Z}) = 0$ for $i > 0$ but for some $i > 0$ $H_i(X; \mathbb{Z}/2) \neq 0$? Prove it or give an example.