DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

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Instructions: Do all problems from section A and all problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. To pass the exam you need to provide **complete** solution to 3 problems from **each** part.

A. Answer all of the following questions.

1. Let S^1 denote the unit circle in $\mathbb{C}$ and define $f : S^1 \times S^1 \to \mathbb{R}$ by $f(z, w) = Re(z) + Re(w)$. Show that f is a Morse function and compute the indices of the critical points.

2. Give an example of a 1-form on $\mathbb{R}^2 - \{(0, 0), (1, 0)\}$ which is closed but not exact and prove both properties.

3. Let $S \subset \mathbb{R}^3$ be a (nonempty) smooth surface. Prove that there exists a plane $ax+by+cz = d$ in $\mathbb{R}^3$ whose intersection with S is a collection of circles and lines.

4. Let Δ be a plane field on a manifold M and let X, Y be two vector fields tangent to Δ. Show that if for some $p \in M$ we have $Y_p = 0$ then $[X, Y]_p \in \Delta$.

5. Prove that every manifold admits a Riemannian metric. Use partitions of unity and not embedding into $\mathbb{R}^N$.

6. Regard $SO(n)$ as a subset of the space of $n \times n$ matrices, which in turn is identified with $\mathbb{R}^{n^2}$. Show that $SO(n)$ is a submanifold and identify the tangent space to $SO(n)$ at the identity I (i.e. the Lie algebra of $SO(n)$).

B. Answer all of the following questions.

7. Let $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ and $S^1 \subset S^3$ is given by $w = 0$. Let X be the quotient space of S^3 obtained by collapsing S^1 to a point. Use any method to compute $H_k(X; \mathbb{Z})$ for all $k \in \mathbb{Z}$.

8. Let S_g denote the closed oriented surface of genus g. Show that if $h < g$ then every map $S_h \to S_g$ has degree 0. Does every such map have to be null-homotopic?

9. Define the concepts of chain maps and chain homotopies, and prove that chain homotopy is an equivalence relation.

10. Let X be a path connected, locally path connected, semi-locally simply connected Hausdorff space. One of the main theorems of covering space theory asserts that there is a simply connected space $\tilde{X}$ and a covering map $p : \tilde{X} \to X$. Outline the construction of $\tilde{X}$ and p as follows:
(a) Describe $\tilde{X}$ as a set.
(b) Describe the topology on $\tilde{X}$.
(c) Describe p. You do not need to prove that p is a covering map.
(d) Outline the proof that $\tilde{X}$ is simply connected.

11. Let $p : \tilde{X} \to X$ be a covering map between cell complexes. Assume that $\tilde{X}$ is connected and that p is null-homotopic. Prove that $\tilde{X}$ is contractible.

12. Give an example of a space X such that $H_i(X; \mathbb{Z}/2) = 0$ for $i > 0$ but for some $i > 0$ $H_i(X; \mathbb{Z}) \neq 0$. Is it possible to have a space X with $H_i(X; \mathbb{Z}) = 0$ for $i > 0$ but for some $i > 0$ $H_i(X; \mathbb{Z}/2) \neq 0$? Prove it or give an example.