A. Answer all of the following questions.

1. Give a nowhere vanishing vector field on $SL_n(\mathbb{R})$.

2. Suppose $M$ is a smooth manifold with a smooth plane field. If the plane field is integrable, then prove that the bracket of any pair of smooth vector fields on $M$ that are tangent to the plane field is another smooth vector field on $M$ that is tangent to the plane field.

3. Give an example of a smooth plane field on a manifold that is not integrable.

4. Given a Lie group $G$ with Lie algebra $\mathfrak{g}$ and some $v \in \mathfrak{g}$, let $\{\theta_v^t\}_{t \in \mathbb{R}}$ be the corresponding 1-parameter group of diffeomorphisms of $G$. Prove that $\theta_v^t(gh) = g\theta_v^t(h)$ for any $t \in \mathbb{R}$ and $g, h \in G$.

5. Prove that there are uncountably many 3-dimensional foliations on a 5-dimensional torus.

6. Let $U$ be a connected Lie subgroup of $GL_n(\mathbb{R})$. Suppose that every element of $U$ has all of its entries below the main diagonal equal to $0$, and all of its entries on the main diagonal equal to $1$. Why is $U$ diffeomorphic to $\mathbb{R}^k$ for some $k$?

7. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Given a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, prove there is a Lie subgroup $H \leq G$ whose Lie algebra is $\mathfrak{h}$. (The entire proof with all details would be best. If not, then try to write the main ideas of the proof.)

8. Let $X$ be a smooth vector field on $S^2$. Prove that $X(p) = 0$ for some $p \in S^2$.

B. Answer all of the following questions.

9. Let $X$ be a topological space and $x_0 \in X$ a basepoint.
   
   (a) Define $\pi_1(X, x_0)$ (describe it as a set and define the group operation; you don’t have to prove that it is well defined or that it is a group).
   
   (b) If $X$ is path-connected and $x_1 \in X$ prove that $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ (write down an explicit isomorphism and check that it works).

10. (a) Define a covering space $p : \tilde{X} \to X$.

    (b) Let $p : \tilde{X} \to X$ be a covering space, $\tilde{x}_0 \in \tilde{X}$, $x_0 \in X$, and $p(\tilde{x}_0) = x_0$. Show that $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective (carefully state the lifting property you are using).
11. Let $X$ be the cell complex obtained from the circle $S^1$ by attaching two 2-cells $e_2^2$ and $e_3^2$ with attaching maps of degrees 2 and 3 respectively. Compute the fundamental group of $X$. Carefully state any theorems you are using.

12. (a) Define the concepts of chain morphisms and chain homotopies.
(b) Show that every map $S^2 \to T^2$ from the 2-sphere to the 2-torus is null-homotopic.
(c) Write down explicit cellular chain complexes $C(S^2)$ and $C(T^2)$ for $S^2$ and $T^2$ and a chain morphism $\Phi : C(S^2) \to C(T^2)$ which is not chain homotopic to a chain morphism representing a constant map.

13. Let $X$ be the space obtained from two $n$-spheres by identifying them along their equatorial $(n - 1)$-spheres. Using any method you like compute $H_i(X)$ for all $i$. State any theorems you are using (e.g. Mayer-Vietoris).

14. (a) State the universal coefficient theorem for cohomology.
(b) Suppose $X$ is a space such that $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}/2\mathbb{Z}$ and $H_i(X) = 0$ for $i > 1$. Compute $H^i(X; \mathbb{Z}/2\mathbb{Z})$ for all $i$. 
