UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

January 3, 2020.

Instructions: This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented.
All problems are worth 20 points. A passing score is 72.

A. Ordinary Differential Equations: Do three problems for full credit

A1. The goal of this problem is to use Picard iterates to find the solution of the initial value problem

$$\frac{dx}{dt} = x^2, \quad x(0) = 1 \tag{1}$$

(a) State a theorem, without proof, with necessary conditions on $f(x,t)$, guaranteeing the existence of a unique solution of the differential equation $\frac{dx}{dt} = f(x,t)$, $x(0) = x_0$.

(b) Verify that the function $f = x^2$ satisfies these necessary conditions.

(c) Define the Picard iterates for the equation (1) and calculate the first three functions $x_0(t)$, $x_1(t)$, and $x_2(t)$.

(d) Prove by induction that

$$x_n(t) = 1 + t + t^2 + \cdots + t^n + O(t^{n+1}) = \sum_{j=0}^{n} t^j + O(t^{n+1}). \tag{2}$$

Hint: You will need to show that $\sum_{j=0}^n t^j \sum_{k=0}^n t^k = \sum_{j=0}^n j t^j + O(t^{n+1})$.

(e) Find the exact solution of (1). How does it compare to the approximate solution found in the previous part?

A2. Consider the first order differential equation

$$\frac{dx}{dt} = f(x,t), \tag{3}$$

where $f(x,t)$ is periodic in t, $f(x,t+1) = f(x,t)$ for all x and t.

(a) Define the Poincare map for this differential equation.

(b) Suppose $f(x,t) = x(A(t) - x)$, and $0 < \alpha < A(t) < \beta$. Prove that the differential equation has at least one nontrivial periodic solution.

A3. (a) For a system of equations $\frac{dx}{dt} = A(t)x$, where $x \in \mathbb{R}^n$ and $A(t)$ is a real $n \times n$ matrix function which is smooth in t, what is Abel's formula? (State without proof).

(b) For the linear differential equation

$$\frac{d^2x}{dt^2} - p(t)\frac{dx}{dt} + q(t)x = 0 \tag{4}$$

where $p(t) > \alpha > 0$ is a smooth function defined for all t, prove there is at least one solution $x(t)$ that is unbounded as $t \to \infty$.

A4. Under what conditions on the parameters a and b does the boundary value problem

$$u'' + u = \cos(x), \quad u'(0) = a, \quad u'(\pi) = b \tag{5}$$

have a solution?

A5. (a) For a general dynamical system, give the definitions of an *invariant set*, *attracting set* and ω–*limit sets*.

(b) For the system

$$\dot{x} = -f(x,y)x - g(x,y)y, \quad \dot{y} = g(x,y)x - f(x,y)y, \tag{6}$$

where $g(x,y) > 0$ for all x, y, and $f(x,y) = x^2 + y^2 + 2y - 1$, identify all critical points and their stability, and show that this system has an invariant annular region which encloses the origin, and therefore has a nontrivial periodic solution.

B. Partial Differential Equations. Do three problems to get full credit

B1. (a) Show that the wave equation $u_{tt} = c^2 u_{xx}$ $(-\infty < x < \infty,\ t > 0)$ conserves energy $E = \int_{-\infty}^{\infty} [(u_t)^2 + c^2 (u_x)^2] dx$.

(b) Show the uniqueness: If two solutions of the wave equation satisfy the same initial conditions, then the solutions are identical.

B2. Obtain the following symmetry properties for the Cauchy problem for the 1D heat equation

$$u_t = k u_{xx} + f(x,t) \quad (-\infty < x < \infty,\ t > 0), \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x).$$

(a) If the data is *even* in x $[\phi(x) = \phi(-x),\ \psi(x) = \psi(-x),\ f(x,t) = f(-x,t)]$, then the solution is *also even* in x $[u(x,t) = u(-x,t)]$.

(b) If the data is *odd* in x $[\phi(x) = -\phi(-x),\ \psi(x) = -\psi(-x),\ f(x,t) = -f(-x,t)]$, then the solution is *also odd* in x $[u(x,t) = -u(-x,t)]$.

(c) If the data is *periodic* in x $[\phi(x+A) = \phi(x),\ \psi(x+A) = \psi(x),\ f(x+A,t) = f(x,t)]$, then the solution is *also periodic* in x with the same period A $[u(x+A,t) = u(x,t)]$.

B3. (a) Why does the heat/diffusion equation imply *infinite* propagation speed?

(b) Why does the wave equation imply *finite* propagation speed?

B4. Suppose someone treats one of the variables in the Laplace equation as time and tries to solve the evolution problem

$$u_{tt} + u_{xx} = 0 \ (0 < x < l,\ t > 0), \quad u(0,t) = u(l,t) = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x),$$

similar to the wave equation. Show that this problem has no continuous dependence on data, even if the time t belongs to a finite interval $0 < t < T$ (T is a positive constant).

B5. (a) Show the uniqueness of Robin's problem

$$\Delta u = f(\mathbf{x}) \ [\mathbf{x} \in D], \quad \nabla u \cdot \mathbf{n} + a(\mathbf{x}) u = h(\mathbf{x}) \ [\mathbf{x} \in \partial D], \quad [D \text{ is a 3D domain, } a(\mathbf{x}) > 0].$$

(b) When $a \equiv 0$, we have the Neumann problem. Show that it has no solution, unless

$$\int_D f d\mathbf{x} = \oint_{\partial D} h \, dA.$$

Describe physical meaning of this fact in the case of steady heat conduction.

(c) Prove the uniqueness of the Neumann problem up to an additive constant.