A. Ordinary Differential Equations: Do three problems for full credit

A1. The simple pendulum consists of a point particle of mass \( m \) suspended from a fixed point by a massless rod of length \( L \), which is allowed to swing in a vertical plane. If friction is ignored then the equation of motion is

\[
\ddot{x} + \omega^2 \sin x = 0, \quad \omega^2 = \frac{g}{L},
\]

where \( x \) is the angle of inclination of the rod with respect to the downward vertical and \( g \) is the gravitational constant.

(a) Using conservation of energy, show that the angular velocity of the pendulum satisfies

\[
\dot{x} = \pm \sqrt{2(C + \omega^2 \cos x)^{1/2}},
\]

where \( C \) is an arbitrary constant. Express \( C \) in terms of the total energy of the system.

(b) Sketch the phase diagram of the pendulum equation in the \((x, \dot{x})\)-plane. Illustrate the one-parameter family of curves given by part (a) for different values of \( C \). Take \(-3\pi \leq x \leq 3\pi\). Indicate the fixed points of the system and the separatrices - curves linking the fixed points. Give a physical interpretation of the underlying trajectories in the two distinct dynamical regimes \(|C| < \omega^2\) and \(|C| > \omega^2\).

(c) Show that in the regime \(|C| < \omega^2\), the period of oscillations is

\[
T = 4\sqrt{\frac{L}{g}} K(\sin x_0/2),
\]

where \( \dot{x} = 0 \) when \( x = x_0 \) and \( K \) is the complete elliptic integral of the first kind, which is defined by

\[
K(\alpha) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \alpha^2 \sin^2 u}} du.
\]

(d) For small amplitude oscillations, the pendulum equation can be approximated by the linear equation

\[
\ddot{x} + \omega^2 x = 0.
\]

Solve this equation for the initial conditions \( x(0) = A, \dot{x}(0) = 0 \) and sketch the phase-plane for different values of \( A \). Compare with the phase-plane for the full nonlinear equation in part (b).
A2. Suppose $A(t)$ is a real $n \times n$ matrix function which is smooth in $t$ and periodic of period $T > 0$. Consider the linear differential equation in $\mathbb{R}^n$

$$\begin{cases}
\frac{dx}{dt} = A(t)x, \\
x(0) = x_0.
\end{cases}$$

(1)

Let $\Phi(t)$ be the fundamental matrix solution with $\Phi(0) = I$.

(a) Define: Floquet Matrix, Floquet Multiplier and Floquet Exponent. How are these related to $\Phi(t)$? State the necessary and sufficient conditions so that (1) has a nonzero $T$-periodic solution.

(b) Prove that the zero solution is unstable for the system $\dot{x} = A(t)x$, where

$$A(t) = \begin{pmatrix} 1 & 1 \\ 0 & \dot{h}(t)/h(t) \end{pmatrix},$$

and $h(t) = 2 + \sin t - \cos t$.

A3. Consider Mathieu’s equation for a parametric oscillator:

$$\ddot{x} + (\alpha + \beta \cos t)x = 0.$$ 

(a) Suppose that $\alpha \approx 1, \beta \approx 0$. Use a perturbation expansion in $\beta$ to show that the transition curves for Mathieu’s equation are given approximately by

$$\alpha = 1 - \frac{\beta^2}{12}, \quad \alpha = 1 + \frac{5}{12} \beta^2.$$ 

(b) Now suppose that $\alpha \approx 1/4 + \alpha_1 \beta, \beta \approx 0$. In the unstable region near $\alpha = 1/4$, solutions of Mathieu’s equation are of the form

$$c_1 e^{\sigma t}q_1(t) + c_2 e^{\sigma t}q_2(t)$$

where $\sigma$ is real and positive, and $q_1, q_2$ are $4\pi$-periodic. Derive the second order equation for $q_1, q_2$ and perform a power series expansion in $\beta$ to show that $\sigma \approx \pm \beta \sqrt{1/4 - \alpha_1^2}$.

(c) Use part (b) to deduce that solutions of the damped Mathieu equation

$$\ddot{x} + \kappa \dot{x} + (\alpha + \beta \cos t)x = 0,$$

where $\kappa = \kappa_1 + O(\beta^2)$, are stable if to first order in $\beta$,

$$\alpha < \frac{1}{4} - \frac{\beta}{2} \sqrt{1 - \kappa_1^2} \text{ or } \alpha > \frac{1}{4} + \frac{\beta}{2} \sqrt{1 - \kappa_1^2}.$$ 

A4. Consider a linear chain of $2N$ atoms consisting of two different masses $m, M$ with $M > m$, placed alternately. The atoms are equally spaced with lattice spacing $a$ with nearest neighbor interactions represented by Hookean springs with spring constant $\beta$. Label the light atoms by even integers $2n, n = 0, ..., N-1$ and the heavy atoms by odd integers $2n-1, n = 1, ..., N$. Denoting their displacements from equilibrium by the variables $U_{2n}$ and $V_{2n-1}$ respectively, Newton’s law of motion gives

$$m\ddot{U}_{2n} = \beta [V_{2n-1} + V_{2n+1} - 2U_{2n}]$$
$$M\ddot{V}_{2n-1} = \beta [U_{2n} + U_{2n-2} - 2V_{2n-1}]$$

Assume periodic boundary conditions $U_0 = U_{2N}$ and $V_1 = V_{2N+1}$.
(a) Sketch the configuration of atoms and briefly explain how the dynamical equations arise from Newton’s law of motion.

(b) Assuming a solution of the form

\[ U_{2n} = \Phi e^{2inka} e^{-i\omega t}, \quad V_{2n+1} = \Psi e^{i(2n+1)ka} e^{-i\omega t}, \]

derive an eigenvalue equation for the amplitudes \((\Phi, \Psi)\) and determine the eigenvalues.

(c) Using part (b), show that there are two branches of solution and determine the speed \(w/k\) on the two branches for small \(k\).

A5. (a) Give definitions for the following: invariant set, attracting set, \(\omega\)-limit set.

(b) Determine the invariant sets and the attracting set of the dynamical system

\[
\begin{align*}
\dot{x} &= -y + x(1 - x^2 - y^2) \\
\dot{y} &= x + y(1 - x^2 - y^2) \\
\dot{z} &= \alpha > 0.
\end{align*}
\]

Also sketch the flow.

(c) Describe what happens to the flow if we identify the points \((x, y, 0)\) and \((x, y, 2\pi)\) in the planes \(z = 0\) and \(z = 2\pi\).

(d) By explicitly constructing solutions on the invariant torus \(x^2 + y^2 = 1, 0 \leq z < 2\pi\), show that the torus is only an attractor if \(\alpha\) is irrational.
B. Partial Differential Equations. Do three problems to get full credit

B1. Suppose that the eigenvalues $\lambda_n$ and normalized eigenfunctions $\phi_n(x)$ of the Dirichlet problem for the Laplacian in a bounded domain $D$ are known: that is,
\[ \nabla^2 \phi_n = \lambda_n \phi_n \text{ in } D, \quad \phi_n = 0 \text{ on } \partial D \]
for all integers $n \geq 0$
(a) Using Green’s Theorem show that $\lambda_n < 0$.
(b) Derive the eigenfunction expansion of the Green’s function $G(x, y)$ for the Helmholtz equation
\[ \nabla^2 G + k^2 G = \delta(x - y) \text{ in } D, \quad G = 0 \text{ on } \partial D, \]
assuming that $k^2 + \lambda_n \neq 0$ for all $n \geq 0$.
(c) Now suppose that $k^2 + \lambda_n = 0$ for some $n$. Show how to construct a generalized Green’s function by solving
\[ (\nabla^2 - \lambda_n)G(x, y) = \delta(x - y) + c\phi_n(x)\phi_n(y) \]
via an eigenfunction expansion with a suitable choice of $c$.

B2. Consider the wave equation
\[ u_{tt} - c^2 u_{xx} = 0, \quad t > 0, x \in \mathbb{R}. \]
along with initial conditions
\[ u(x, 0) = g(x), \quad u_t(x, 0) = h(x). \]
(a) Assuming $c$ is constant, derive d’Alembert’s Formula
\[ u(x, t) = \frac{1}{2} [g(x + ct) + g(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} h(\xi) \, d\xi. \]
(b) Determine the solution for the initial data $u(x, 0) = 1$ if $|x| < a$, $u(x, 0) = 0$ if $x > |a|$;
$u_t(x, 0) = 0$
(c) Determine the solution for the initial data $u(x, 0) = 0$; $u_t(x, 0) = 1$ if $|x| < a$, $u_t(x, 0) = 0$ if $x > |a|$.

B3. Suppose that $\rho(x, t)$ is the number density of cars per unit length along a road, $x$ being distance along the road, such that
\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho(1 - \rho))}{\partial x} = 0. \]
(a) Show that $\rho$ is constant along the characteristics
\[ \frac{dx}{dt} = 1 - 2\rho, \]
and derive the following Rankine-Hugoniot condition for the speed of a shock $x = S(t)$:
\[ \frac{dS}{dt} = \frac{[\rho(1 - \rho)]^+}{[\rho]^+}. \]
(b) A queue is building up at a traffic light $x = 1$ so that, when the light turns to green at $t = 0$, 

$$
\rho(x, 0) = \begin{cases} 
0, & \text{if } x < 0 \text{ and } x > 1; \\
x, & \text{if } 0 < x < 1.
\end{cases}
$$

Solve the corresponding characteristic equations, and sketch the resulting characteristic curves. Deduce that a collision first occurs at $x = 1/2$ when $t = 1/2$, and that thereafter there is a shock such that 

$$
\frac{dS}{dt} = \frac{S + t - 1}{2t}.
$$

B4. Suppose that $u(x)$ is a $C^2$ harmonic function in the domain $\Omega \subset \mathbb{R}^n$, so $\Delta u = 0$ in $\Omega$.

(a) Prove the mean value property: if $x \in \Omega$ and $r > 0$ is chosen such that $B_r(x) \subset \Omega$ (ball of radius $r$ centered at $x$) then 

$$
u(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} u(s) ds,
$$

where $\omega_n$ is the measure of $\partial B_1$. Hence show that 

$$
u(x) \leq \frac{n}{\omega_n r^n} \int_{B_r(x)} u(y) dy.
$$

(b) Assuming $\Omega$ is connected, prove that $u$ can attain its maximum value at an interior point $x \in \Omega$, only if $u$ is constant.

B5. Consider the equation 

$$
\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = \epsilon \frac{\partial^2 u}{\partial x^2},
$$

for $-\infty < x < \infty$ and $t > 0$.

(a) Look for a traveling wave solution $u(x, t) = U(z)$, $z = (x - Vt)/\epsilon$, with velocity $V > 0$ and $U(z) \rightarrow U_\pm$ as $z \rightarrow \pm \infty$. Solve the resulting ODE for $U(z)$ and deduce that 

$$V = \frac{[U^2/2]_{-\infty}^{\infty}}{[U]_{-\infty}^{\infty}}.
$$

(b) Discuss how the traveling wave solution relates to shock solutions of the quasilinear equation obtained by setting $\epsilon = 0$.

(c) Using phase-plane analysis, show that the wave profile $U$ is monotonically decreasing, that is, $dU/dz < 0$. 

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