UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Applied Mathematics

January 4th, 2017.

Instructions: This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are weighted equally.

Part A.

A1. If (x_n) is a weakly convergent sequence in a normed space X, say, $x_n \xrightarrow{w} x_0$, show that there is a sequence (y_n) of linear combinations of elements of (x_n) which converges strongly to x_0. Hint: First prove that $x_0 \in \overline{\operatorname{span}(x_n)}$.

A2. Let f be a real-valued and twice continuously differentiable on an interval $[a, b]$. Let $\hat{x}$ be a simple zero of f, *i.e.* $f'(\hat{x}) \neq 0$. Use the Banach Fixed Point Theorem to show that *Newton's method*, defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converges to $\hat{x}$ for any x_0 sufficiently close to $\hat{x}$.

A3. Consider the multiplication operator $T\colon L^2[0,1] \to L^2[0,1]$ defined by $Tx(t) = tx(t)$.

(a) Show that T is a bounded, self-adjoint linear operator.

(b) Find the spectrum $\sigma(T)$, point spectrum $\sigma_p(T)$, continuous spectrum $\sigma_c(T)$, residual spectrum $\sigma_r(T)$, and resolvent $\rho(T)$.

(c) Show that the corresponding spectral family $(E_\lambda)_{\lambda\in\mathbb{R}}$ is given by

$$E_\lambda x = \begin{cases} 0 & \text{if } \lambda < 0 \\ v_\lambda & \text{if } 0 \le \lambda \le 1 \\ x & \text{if } \lambda > 1 \end{cases} \qquad \text{where} \quad v_\lambda(t) = \begin{cases} x(t) & \text{if } 0 \le t \le \lambda \\ 0 & \text{if } \lambda < t \le 1. \end{cases}$$

A4. Let X, Y be Hilbert spaces and let $(e_n) \subset X$ be a total orthonormal sequence. Recall that an operator $T\colon X \to Y$ is said to be *Hilbert-Schmidt* if the Hilbert-Schmidt norm $\|T\|_{\mathrm{H}S} := \left(\sum_{n=1}^{\infty} \|Te_n\|^2\right)^{\frac{1}{2}}$ is finite.

(a) Show that the Hilbert-Schmidt norm is independent of the sequence (e_n) chosen. Hint: Use Parseval's identity.

(b) Show that a Hilbert-Schmidt operator is compact.

(c) For a Hilbert-Schmidt operator, T, show that $\|T\| \le \|T\|_{\mathrm{H}S}$, where $\|\cdot\|$ denotes the operator norm.

A5. Let $T\colon H \to H$ be a compact, self-adjoint linear operator defined on a separable Hilbert space H. Let (v_n) be a complete orthonormal sequence of eigenvectors of T and let (λ_n) be the corresponding eigenvalues.

(a) Let $\lambda \in \rho(T)$. Show that the resolvent of T at λ, acting on some $x \in H$ can be written as

$$R_\lambda x = \sum_{n=1}^{\infty} \frac{1}{\lambda_n - \lambda} \langle x, v_n \rangle v_n.$$

(b) Use part (a) to show that if $\lambda, \mu \in \rho(T)$ then the first resolvent identity holds:

$$R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu.$$

Part B.

B1. (a) Find all values of i^i.

(b) Find all solutions of the equation $\cos z = 2$.

(c) Find all solutions of the equation $\tan z = i$.

B2. The following function $f(x)$ [x is a real variable] can be represented by a series (Taylor or Laurent) in powers of x. Find the radius of convergence of the series in each case

$$\begin{array}{llcl}
\text{(a)} & f(x) & = & \left(\dfrac{\sin x}{x}\right)^2, \\
\text{(b)} & f(x) & = & \dfrac{\sin x}{x^2}, \\
\text{(c)} & f(x) & = & \left(\dfrac{x}{\sin x}\right)^2, \\
\text{(d)} & f(x) & = & \dfrac{x^2}{\sin x}, \\
\text{(e)} & f(x) & = & \dfrac{\sin x}{x^2+1},
\end{array}$$

[You do not need to find the series themselves.]

B3. Integrate

$$\begin{array}{ll}
\text{(a)} & \displaystyle\int_0^{2\pi} \frac{d\phi}{(p + q\cos\phi)^2} \quad (\text{parameters } p > q > 0), \\
\text{(b)} & \displaystyle\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1 + e^x}\, dx \quad (\text{parameter } \alpha \in (0,1)), \\
\text{(c)} & \displaystyle\int_0^{\infty} \sin(a^2 x^2)\, dx \quad (\text{positive parameter } a).
\end{array}$$

B4. Prove that if all singularities of an analytic function $f(z)$ in the extended complex plane $\overline{\mathbb{C}}$ are poles, then $f(z)$ is a rational function (i.e. the ratio of two polynomials).

B5. Prove that a bilinear map takes all circles into circles.