PhD Preliminary Qualifying Examination
Applied Mathematics
January 7 2015

Instructions: Answer three questions from Part A and three questions from Part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Let $M$ be a subset of a Hilbert space $H$. The goal of this problem is to show that:
   \[ \overline{\text{span } M} = H \text{ if and only if } M^\perp = \{0\}. \]
   (a) Assume $\overline{\text{span } M} = H$ and let $x \in M^\perp$. Show that $x = 0$.
   (b) Assume $M^\perp = \{0\}$. Show that $\overline{\text{span } M} = H$.

2. Let $(x_n)$ be a sequence in a Hilbert space $H$. Show that:
   \[ x_n \to x \text{ strongly if and only if } (x_n \to x \text{ weakly and } ||x_n|| \to ||x||). \]

3. Let $u, v$ be non-zero elements of a Hilbert space $H$. Consider the linear operator $T : H \to H$ defined for $x \in H$ by $Tx = u \langle x, v \rangle$.
   (a) Briefly explain why the operator $T$ is compact.
   (b) Find a condition on $u$ and $v$ that guarantees that the equation
       \[ (I - T)x = y \]
       admits a unique solution $x$ for all $y \in H$.

4. Let $(\lambda_n) \in \ell^2$ and consider the operator $T : \ell^2 \to \ell^2$ defined by $y = Tx$, where $x = (\xi_j) \in \ell^2$, $y = (\eta_j) \in \ell^2$ and
   \[ \eta_j = \sum_{k=1}^{\infty} \alpha_{jk} \xi_k, \quad j = 1, 2, \ldots, \]
   where the $\alpha_{jk}$ satisfy
   \[ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_{jk}|^2 < \infty. \]
   Show that $T$ is a compact operator. \textbf{Hint:} Approximate $T$ with finite rank operators $T_n$.

5. Let $T : X \to X$ be a bounded linear operator defined on a Banach space $X$. Denote by $R_\lambda$ the resolvent operator associated with $T$ for some $\lambda \in \rho(T)$. The resolvent set of $T$ is denoted by $\rho(T)$.
   (a) Use Neumann series to show that for $\lambda, \mu \in \rho(T)$,
       \[ R_\lambda = \sum_{j=0}^{\infty} (\lambda - \mu)^j R_\mu^{j+1}, \]
       where the series is absolutely convergent in the operator norm when
       \[ |\lambda - \mu| < ||R_\mu||^{-1}. \]
   (b) Deduce from part (a) whether the spectrum $\sigma(T)$ is open, closed or neither.
Part B.

1. Find the radius of convergence of the Taylor series with the center at the origin \((x_0 = 0)\) for the following function

\[
\begin{align*}
(a) & \quad f(x) = \frac{\sqrt{x + 3}}{x^2 + 3} \\
(b) & \quad f(x) = \frac{1}{(\cos x + 3)^2}
\end{align*}
\]

\((x)\) is a real variable; you do not need to find the series themselves).

2. Let \(D_1\) and \(D_2\) be two disjoint domains, whose boundaries share a common curve \(\Gamma\). Let

- \(f(z)\) be analytic in \(D_1\) and continuous in \(D_1 \cup \Gamma\)
- \(g(z)\) be analytic in \(D_2\) and continuous in \(D_2 \cup \Gamma\)
- \(f(z) = g(z)\) for \(z \in \Gamma\)

Show that the function

\[
H(z) = \begin{cases} 
  f(z) & z \in D_1 \\
  f(z) = g(z) & z \in \Gamma \\
  g(z) & z \in D_2
\end{cases}
\]

is analytic in \(D = D_1 \cup \Gamma \cup D_2\).

[Hint: Use the Morera theorem.]

3. (a) Show that any analytic function (not identically equal to zero) can have only isolated zeros inside its analyticity domain.

(b) Can an analytic function have a non-isolated singularity?

(c) Prove the Uniqueness Theorem: If two functions are analytic in a domain \(D\) and equal on some set of points that has a limiting point inside \(D\), then these functions are identically equal in \(D\).

4. Evaluate

(a) \(\int_{-\infty}^{\infty} e^{iz^2} \, dx\)

(b) \(\int_{0}^{\infty} \frac{\sin x}{x} \, dx\).

5. Find the leading behavior, as \(s \to +\infty\), of the integral

(a) \(I(s) = \int_{0}^{3} \frac{1}{\sqrt{x^2 + 2x}} e^{-sx} \, dx\)

(b) \(I(s) = \int_{0}^{\pi/2} e^{is \cos x} \, dx\)