PhD Preliminary Qualifying Examination
Applied Mathematics

January, 2014

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Consider the problem $u''(x) = x\delta'(x)$, where δ is the "delta function". Explain what it means for this problem to have a solution u in the sense of distributions. Find a distributional solution u and prove that it does in fact solve the problem in the sense of distributions.

2. Let A be a real $m \times n$ matrix, and let $b \in \mathbb{R}^m$ be a column vector. Consider the problem

$$A^*Ax = A^*b. \quad (*)$$

   (a) Explain the relationship between $(*)$ and the problem

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|^2.$$

   (b) Under what conditions on A, A^*, and b is it guaranteed that $(*)$ has a solution x? Under what conditions is it guaranteed that $(*)$ has a unique solution?

   (c) Given $\epsilon > 0$, consider the problem

$$(A^*A + \epsilon I)x_\epsilon = A^*b. \quad (**)$$

   Formulate an associated minimization problem, describe existence and uniqueness, and explain how $\lim_{\epsilon\to 0^+} x_\epsilon$ is related to solutions of $(*)$.

3. Consider the problem $u''(x) = f(x)$, on $(0, 1)$, with boundary conditions $u'(0) = u'(1) = 0$.

   (a) Write down the weak formulation of the problem.

   (b) Using a piecewise linear finite element basis with uniform grid size h, use the Galerkin method to find a finite-dimensional (matrix) approximate representation for the problem. Do not solve the equations.

   (c) Neither of the problems (a) and (b) have unique solutions in general. Explain why not, and describe what can be done to correct the deficiency.

4. Define $Ku(x) = \displaystyle\sum_{n=0}^{10} \left(\int_{-1}^{1} e^{-ny} u(y)\, dy \right) x^n$, for $x \in [-1, 1]$.

   (a) Find the adjoint K^* of the operator K.

   (b) Assuming $\lambda \neq 0$ is fixed and real, use the Fredholm Alternative to characterize the set of functions g such that the problem $(K - \lambda I)u = g$ on $[-1, 1]$ has a solution.

   (c) From the structure of the operator K, what can be said about the set $\sigma_p = \{\lambda \in \mathbb{C} : (K - \lambda I)u = 0 \text{ has a solution } u \neq 0\}$? There is no need to actually calculate σ_p.

5. Construct the Green's function for the problem

$$u''(x) + \alpha^2 u(x) = f(x), \text{ on } (0, 1), \quad u(0) = u(1) = 0.$$

   Write the solution u in terms of the Green's function, and determine values of α for which the Green's function does not exist.

Part B.

1. The function $f(z)$ is analytic in the extended complex plane except for two points $z = 0$ and $z = 2$, where it has simple poles. It is known that

$$\oint_{|z|=1} f(z)dz = 1, \qquad \oint_{|z|=3} f(z)dz = 0.$$

   Find $f(z)$ (unique up to an arbitrary additive constant).

2. Find the radius of convergence of the Taylor series

$$\text{for the function } f(x) = \frac{1}{\cos^2 x + 4} \quad \text{around the origin } x_0 = 0.$$

   {You do not need to find the Taylor series itself.}

3. Evaluate the integral

$$\int_0^\infty e^{ix^2} dx.$$

4. Find the leading behavior of the integral

$$I = \int_0^3 \frac{\cos x}{\sqrt{x}} e^{-sx}\, dx$$

   for large positive s.

5. Formulate and prove Morera's theorem.