PhD Preliminary Qualifying Examination:
Applied Mathematics
Jan. 3, 2013

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. The \( n \) data points \((x_i, y_i), i = 1, 2, \ldots, n\) are believed to lie on an exponential curve \( y_i = A \exp(\lambda x_i) \). Estimate \( \lambda \).

2. (a) Specify the weak formulation for the differential equation

\[
    u'' + \lambda \delta(x - \frac{1}{2})u = f(x)
\]

subject to boundary conditions \( u(0) = u(1) = 0 \).

(b) Find all values of \( \lambda \) for which this differential equation with \( f(x) = 0 \) has a nontrivial solution and verify that the corresponding solution is a weak solution.

(c) Suppose \( \lambda \) is such that this differential equation with \( f(x) = 0 \) has only the trivial solution. Find the solution with \( f(x) = 1 \).

3. (a) Find all conditions on \( \alpha \), \( \beta \), \( \gamma \), and \( f(x) \) for which solutions of

\[
    u'' + \alpha^2 u = f(x), \quad u(0) = \beta, \quad \alpha u'(0) - u'(1) = \gamma
\]

exist.

(b) Find an integral representation for the solution in the case that \( \alpha = 0 \).

4. Consider the integral equation

\[
    \phi(x) - \lambda \int_0^\pi \sin(x + t)\phi(t)dt = \cos(x) + \sin(x), \quad 0 \leq x \leq \pi.
\]

(a) Find the unique solution when \( \lambda \neq \pm 2/\pi \). Why is a unique solution guaranteed to exist?

(b) Show that there is no solution when \( \lambda = 2/\pi \).
(c) Show that when $\lambda = -2/\pi$ there is a one-parameter family of solutions of the form

$$\phi(x) = f_p(x) + \alpha f_h(x).$$

Determine $f_h(x)$.

5. (a) Describe the Gram-Schmidt orthogonalization procedure for a set of $n$-vectors $\{u_1, u_2, \ldots, u_k\}$.

(b) Suppose $k < n$ and that the vectors $\{u_1, u_2, \ldots, u_k\}$ are linearly independent. Show that the Gram-Schmidt procedure is equivalent to factoring the matrix $U$ with column vectors $\{u_1, u_2, \ldots, u_k\}$ as $U = QR$ where $R$ is triangular. What is the structure of $Q$?

(c) Use this factorization of $U$ to find the least-squares solution of $Ux = b$. 
Part B.

1. (a) Formulate and derive the *argument principle* [which determines the difference between the number of zeros \((N_0)\) and poles \((N_\infty)\) of an analytic function \(f(z)\)].

(b) Applying this principle, determine the number of zeros located inside the first quadrant \(\{z = x + iy : x > 0, y > 0\}\) of the function \(f(z) = z^5 + 1\).

2. Find the image of the half-strip \(\{z = x + iy : x > 0, 0 < y < 1\}\) under the mapping \(w = 1/z\).

3. Calculate the integral

\[
I = \int_{0}^{\infty} \frac{x^\alpha}{1 + x} \, dx
\]

where \(\alpha\) is a real number for which the integral converges.

4. Formulate and derive the *uncertainty principle*. Is its inequality optimal?

5. Find at least three terms of the asymptotic expansion of the integral

\[
I(s) = \int_{0}^{1} \ln t \ e^{ist} \, dt, \quad s \text{ is real and } s \to +\infty.
\]