PhD Preliminary Qualifying Examination:  
Applied Mathematics  
Jan. 7, 2010

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Let

\[ A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \]

(a) Find the eigenvalues and eigenvectors of \( A \), and the range of the function

\[ \phi(x) = \frac{\langle Ax, x \rangle}{\langle x, x \rangle} \]

where \( x = (x_1, x_2, x_3) \) is a real-valued vector.

(b) Compute \( \exp(A) \).

(c) Consider the equation

\[ Ax = \mu x \]

Find all solutions for \( \mu = 1 \) and for \( \mu = 3 \).

2. Consider linear operator

\[ H = -\frac{d^2}{dx^2}, \]

acting on \( \psi(x) \in L^2[0, \pi] \) with periodic boundary conditions \( \psi(0) = \psi(\pi) = 0 \). Find the eigenvalues and eigenfunctions of \( H \). Show that the eigenvectors of \( H \) corresponding to distinct eigenvalues are orthogonal. Are the eigenvectors of \( H \) complete, and if so, in what sense?

(b) Solve the equation

\[ H \psi(x) = x^2, \]

representing the solution as an expansion in eigenfunctions of \( H \).

3. Let

\[ f(x) = \begin{cases} \frac{100}{n}, & x \ \text{rational}, \ x = \frac{m}{n} \\ -2, & x \ \text{irrational}. \end{cases} \]
Does the Riemann integral of \( f(x) \) over the interval \([-1, 1]\) exist? If so, compute it.

Does the Lebesgue integral of \( f(x) \) over the interval \([-1, 1]\) exist? If so, compute it.

Why is the Lebesgue integral used in defining the Hilbert space \( L^2[0, 1] \), and not the Riemann integral?

4. Let

\[ f(x) = \text{signum}(x) = \begin{cases} 
1, & x > 0 \\
0, & x = 0 \\
-1, & x < 0 
\end{cases} \]

(a) Find its first and second derivatives using the theory of distributions.

(b) Compute

\[ I_0 = \int_{-1}^{\infty} \log (x + 10) f(x) \, dx, \]
\[ I_1 = \int_{-1}^{\infty} \log (x + 10) f'(x) \, dx, \]
\[ I_2 = \int_{-1}^{\infty} \log (x + 10) f''(x) \, dx. \]

5. Using Green’s functions, solve the problem

\[ \frac{d^2 u}{dx^2} = \frac{1}{1 + x^2}, \quad u(-1) = u(1) = 0 \]

for \( u(x) \), \( x \in [-1, 1] \) (obtain an integral representation for the solution, but do not evaluate the integral).
Part B.

1. Find a solution \( u(x, y) \) of Laplace’s equation on the domain \(-\infty < x < \infty, 0 < y < \infty\) for which \( u(x, 0) = x^{1/2} \) for \( 0 < x < \infty \). What is \( u(x, 0) \) for \(-\infty < x < 0\)?

2. Use Jordan’s Lemma (and describe how Jordan’s Lemma is used) to evaluate the integral

\[
I = \int_0^\infty \frac{\cos ax}{x^2 + 1} \, dx.
\]

3. Use Fourier transforms to solve the integral equation

\[
\int_{-\infty}^{\infty} k(x-y)u(y)\,dy - u(x) = f(x)
\]

with \( k(x) = H(x) \), the Heaviside function.

4. Use the \( z \)-transform to solve the system of equations

\[
\frac{du_n}{dt} = \frac{1}{h^2}(u_{n+1} - 2u_n + u_{n-1}), \quad -\infty < n < \infty
\]

with \( u_n(t = 0) = \sin \frac{2\pi n}{k} \), with \( k \) an integer.

5. Find the leading order term of the asymptotic expansion of the integral

\[
I(s) = \int_{-\infty}^{\infty} e^{is(t + t^3/3)} \, dt, \quad s \text{ is real and } s \to +\infty,
\]

as well as a rigorous estimate of the error.