PhD Preliminary Qualifying Examination:  
Applied Mathematics  
January, 2009

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. (a) The least squares psuedo-inverse of $A$ is a matrix $B$ which satisfies what two properties?
   (b) Find the pseudoinverse of the matrix
   
   $$ A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}. $$

2. The Haar function $\phi(x)$ is defined by
   $$ \phi(x) = \begin{cases} 1 & \text{when } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases} $$

   and the mother Haar wavelet $W(x)$ is defined by
   $$ W(x) = \begin{cases} 1 & \text{when } 0 < x < 1/2 \\ -1 & \text{when } 1/2 < x < 1 \\ 0 & \text{elsewhere}. \end{cases} $$

   Express the function
   $$ f(x) = \begin{cases} 2 & \text{when } 0 < x < 1/4 \\ 0 & \text{when } 1/4 < x < 1/2 \\ -1 & \text{when } 1/2 < x < 3/4 \\ 1 & \text{when } 3/4 < x < 1 \\ 0 & \text{elsewhere} \end{cases} $$

   as a linear combination of 4 orthogonal functions: the Haar function and 3 Haar wavelets
   $$ W_{mn}(x) = 2^{m/2}W(2^m x - n), $$
   where $m$ and $n$ are appropriate integers.
3. (a) Define what is meant by a precompact (or sequentially compact) set and what is meant by a compact linear operator, assuming linear operators and bounded sets have been already been defined.

(b) If $K$ is a compact linear operator and if $\{\phi_n\}$ is an infinite set of orthonormal functions then (using Bessel’s inequality, if needed) show that

$$\lim_{n \to \infty} K\phi_n = 0.$$ 

4. For the differential operator $Lu = u'' + (x^2 - 1)u'$ on the interval $[0, 1]$ with boundary conditions $u(0) = u'(1)$ and $u(1) = u'(0)$ find the adjoint operator and its domain.

5. (a) Write down the set of linear equations which if solved give the Green’s function for the operator $Lu = d^3u/dx^3$ on the interval $[0, 1]$ with boundary conditions $u(0) = u'(0) = u(1) = 0$. (There is no need to explicitly solve these equations).

(b) Letting $G(x, y)$ denote the Green’s function which solves (a), and using a particular solution of $d^3u/dx^3 = 0$, find the general solution to $d^3u/dx^3 = f(x)$ on the interval $[0, 1]$ with boundary conditions $u(0) = u'(0) = 0$ and $u(1) = 1$. 

2
Part B.

1. (a) Verify that the real and imaginary parts of a complex analytic function satisfy Laplace’s equation.

(b) Give an interpretation of the complex function \( w(z) = z + \frac{a^2}{z} \) as a flow past some object. That is, determine the streamlines for this function. Where are the stagnation points?

2. (a) Use contour integration to explicitly evaluate \( f(a) = \int_0^\infty \frac{x \sin x dx}{x^2 - a} \) for complex \( a \). For what values of complex \( a \) is this possible?

(b) Is the function \( f(a) \) an analytic function of \( a \)? Identify all singularities, branch points, branch cuts, etc.

3. The windowed Fourier transform of a function \( f(x) \in L^2(-\infty, \infty) \) is defined by

\[
Gf(\omega, \mu) = \int_{-\infty}^{\infty} g(x - \mu)f(x)e^{-\omega x}d\mu dx.
\]

Suppose \( g(x) \) is a real valued function. State and verify the formula for the reconstruction of \( f(x) \) from \( Gf \), including any additional conditions on \( g(x) \). You may assume the validity of the Fourier transform.

4. (a) Find the spherically symmetric eigenfunctions and corresponding eigenvalues for the Laplacian on a spherical domain of radius \( R \) with Dirichlet boundary conditions \( u(R) = 0 \).

(b) Use these eigenfunctions to solve the heat equation on a spherical domain of radius \( R \), with \( u(R, t) = U_0 \) and \( u(r, 0) = 0 \).

(c) Estimate the time \( t \) at which \( u(0, t) \) is \( \frac{U_0}{2} \). How does this time depend on the radius \( R \)?

5. (a) Find the first term of the asymptotic representation of

\[
n! = \int_0^\infty \exp(-t)t^{a-1}dt
\]

for large \( n \). (This approximation is called Stirling’s formula.)

(b) What is the order of the error term and how can this be rigorously justified? (Invoke the appropriate theorem.)