PhD Preliminary Qualifying Examination
Applied Mathematics
August, 2013

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Suppose that $E$ is a bounded linear operator on a Hilbert space $H$, with operator norm $\|E\| = r < 1$. Show that the operator $I - E$, where $I$ is the identity, is bounded and invertible. Give an estimate for $\|(I - E)^{-1}\|$, and a convergent series representation for $(I - E)^{-1}$.

2. Suppose that the square real matrix $A = (a_{ij})$ satisfies $a_{ij} = a_{ji}$ for all $i, j$.

   (a) Show that every eigenvalue $\lambda$ of $A$ must be real.

   (b) Show that for any eigenpairs $(\lambda_1, u_1)$, $(\lambda_2, u_2)$, with $\lambda_1 \neq \lambda_2$, the eigenvectors $u_1$ and $u_2$ must be orthogonal.

   (c) Let $c = \min_{\|u\| = 1} u^T A u$. Prove that if $c > 0$, then $A^{-1}$ exists. In this case, find an estimate for the matrix operator norm $\|A^{-1}\|$.

3. Consider the problem $Lu = 1$ on $(0, \pi)$, where $Lu = u'' + u$ with boundary conditions $u'(0) = u'(\pi) = 0$.

   (a) Write down the weak formulation of the problem.

   (b) Find the $L^2[0, \pi]$ adjoint of $L$, and determine whether or not $L$ is self-adjoint.

   (c) Find the nullspace of $L$. Use Fredholm theory to characterize the set of functions $f$ such that $Lu = f$ has a solution. Verify that $f = 1$ satisfies your condition, and find an infinite set of solutions to the problem $Lu = 1$. 
4. Define the operator $K$ on $L^2[0, 1]$ by $Ku(x) = \int_0^1 k(x, y)u(y) \, dy$, for $x \in [0, 1]$, where $k(x, y)$ is a real-valued, bounded, continuous function, with $k(x, y) > 0$.

(a) Find the $L^2$ adjoint $K^*$ for $K$, and find a condition on $k$ which guarantees that $K$ is self-adjoint.

(b) Describe the properties of the set

$$\{ \lambda \in \mathbb{C} : (K - \lambda I)u = 0 \text{ for some } u \neq 0 \}$$

(cardinality, bounds, etc.).

(c) Consider the problem $Ku = f$. Let $X$ be the set of all $f \in L^2[0, 1]$ such that a unique solution $u = K^{-1}f$ exists. Can we be assured that there is a constant $C$ such that $\|K^{-1}f\| \leq C\|f\|$, for all $f \in X$? Why or why not?

5. Let $f(x) = |x| - 1$ on the interval $[-1, 1]$.

(a) Treating $f$ as a distribution, calculate

$$\int_{-1}^1 f'(x)f'(x) \, dx \text{ and } \int_{-1}^1 f(x)f''(x) \, dx.$$ 

(b) Explain why $\int_{-1}^1 f'(x)f''(x) \, dx$ and $\int_{-1}^1 f(x)f'''(x) \, dx$ do not make sense.

(c) In general, what property is required of $g(x)$ so that $\int_{-1}^1 g(x)f^{(n)}(x) \, dx$ is well-defined for a given $n^{\text{th}}$ derivative?
Part B.

1. Find the radius of convergence of the Taylor series for the function
   \[ \frac{1}{2 + \cos x} \] (\(x\) is a real variable)
   around the origin (\(x_0 = 0\)).
   {You do not need to find the Taylor series itself.}

2. (a) Show that any bilinear map takes circles into circles.
   (b) Find the image of the triangle
   \[ \{z = x + iy : x \geq 0, \ y \geq 0, \ x + y \leq 1\} \]
   under the mapping \(w = 1/z\).

3. Evaluate the integrals:
   (a) \(I = \int_0^\infty \frac{\sin x}{x} \, dx\),
   (b) \(I = \int_0^\infty \frac{x^\alpha}{2+x} \, dx\) \((-1 < \alpha < 1)\).


5. Find the leading behavior, as \(s \to +\infty\), of the integral
   (a) \(I = \int_0^3 \frac{1}{\sqrt{x^2 + 2x}} e^{-sx} \, dx\),
   (b) \(I = \int_0^{\pi/2} e^{is\cos x} \, dx\).