PhD Preliminary Qualifying Examination
Applied Mathematics

August, 2013

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Suppose that E is a bounded linear operator on a Hilbert space H, with operator norm $\|E\| = r < 1$. Show that the operator $I - E$, where I is the identity, is bounded and invertible. Give an estimate for $\|(I-E)^{-1}\|$, and a convergent series representation for $(I-E)^{-1}$.

2. Suppose that the square real matrix $A = (a_{ij})$ satisfies $a_{ij} = a_{ji}$ for all i, j.

   (a) Show that every eigenvalue λ of A must be real.

   (b) Show that for any eigenpairs (λ_1, u_1), (λ_2, u_2), with $\lambda_1 \neq \lambda_2$, the eigenvectors u_1 and u_2 must be orthogonal.

   (c) Let $c = \min\limits_{\|u\|=1} u^T A u$. Prove that if $c > 0$, then A^{-1} exists. In this case, find an estimate for the matrix operator norm $\|A^{-1}\|$.

3. Consider the problem $Lu = 1$ on $(0, \pi)$, where $Lu = u'' + u$ with boundary conditions $u'(0) = u'(\pi) = 0$.

   (a) Write down the weak formulation of the problem.

   (b) Find the $L^2[0, \pi]$ adjoint of L, and determine whether or not L is self-adjoint.

   (c) Find the nullspace of L. Use Fredholm theory to characterize the set of functions f such that $Lu = f$ has a solution. Verify that $f = 1$ satisfies your condition, and find an infinite set of solutions to the problem $Lu = 1$.

4. Define the operator K on $L^2[0,1]$ by $Ku(x) = \int_0^1 k(x,y)u(y)\,dy$, for $x \in [0,1]$, where $k(x,y)$ is a real-valued, bounded, continuous function, with $k(x,y) > 0$.

   (a) Find the L^2 adjoint K^* for K, and find a condition on k which guarantees that K is self-adjoint.

   (b) Describe the properties of the set
   $$\{\lambda \in \mathbb{C} : (K - \lambda I)u = 0 \text{ for some } u \neq 0\}$$
   (cardinality, bounds, etc..).

   (c) Consider the problem $Ku = f$. Let X be the set of all $f \in L^2[0,1]$ such that a unique solution $u = K^{-1}f$ exists. Can we be assured that there is a constant C such that $\|K^{-1}f\| \leq C\|f\|$, for all $f \in X$? Why or why not?

5. Let $f(x) = |x| - 1$ on the interval $[-1,1]$.

   (a) Treating f as a distribution, calculate
   $\int_{-1}^1 f'(x)f'(x)\,dx$ and $\int_{-1}^1 f(x)f''(x)\,dx$.

   (b) Explain why $\int_{-1}^1 f'(x)f''(x)\,dx$ and $\int_{-1}^1 f(x)f'''(x)\,dx$ do not make sense.

   (c) In general, what property is required of $g(x)$ so that $\int_{-1}^1 g(x)f^{(n)}(x)\,dx$ is well-defined for a given n^{th} derivative?

Part B.

1. Find the radius of convergence of the Taylor series for the function

$$\frac{1}{2+\cos x} \qquad (x \text{ is a real variable})$$

around the origin ($x_0 = 0$).
{You do not need to find the Taylor series itself.}

2. (a) Show that any bilinear map takes circles into circles.

   (b) Find the image of the triangle

$$\{z = x + iy : \qquad x \geq 0, \quad y \geq 0, \quad x + y \leq 1\}$$

   under the mapping $w = 1/z$.

3. Evaluate the integrals:

   (a) $I = \int_0^\infty \frac{\sin x}{x}\, dx$,

   (b) $I = \int_0^\infty \frac{x^\alpha}{2+x}\, dx \qquad (-1 < \alpha < 1)$.

4. Formulate and derive the *uncertainty principle*. Is its inequality optimal? Explain.

5. Find the leading behavior, as $s \to +\infty$, of the integral

   (a) $I = \int_0^3 \frac{1}{\sqrt{x^2+2x}} e^{-sx}\, dx$,

   (b) $I = \int_0^{\pi/2} e^{is\cos x}\, dx$.