^PhD Preliminary Qualifying Examination:

Applied Mathematics

August 18, 2009

INSTRUCTIONS: Answer three questions from Part A and three questions from Part B. Indicate clearly which questions you wish to have graded.

Part A.

1. (a) Let \( A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \). Find the maximum value of the quadratic form \( \langle Ax, x \rangle \) for \( ||x|| \leq 1 \). (b) Prove that eigenvectors corresponding to distinct eigenvalues of a self-adjoint matrix are orthogonal. (c) Let \( H = -\frac{d^2}{dx^2} \) on \( L^2[0,1] \) with boundary conditions \( \psi(0) = \psi(1) = 0 \). Find the eigenvalues and eigenvectors of \( H \). Show that the eigenvectors of \( H \) corresponding to distinct eigenvalues are orthogonal.

2. (a) Define Cauchy sequence, and what it means for a linear space to be complete. (b) Let \( C[0,1] \) be the set of real–valued functions which are continuous on \([0,1]\). Show that \( C[0,1] \) is complete under under the uniform norm \( ||f|| = \sup_{x \in [0,1]} |f(x)| \). (c) Let \( f_n(x) = x^n \) for \( x \in [0,1] \). Find \( f = \lim_{n \to \infty} f_n \). Is \( f(x) \) continuous? Does this violate the completeness you showed in (a)? Explain.

3. State and prove the Riesz Representation Theorem for a Hilbert space.

4. Let \( T : \ell^2 \to \ell^2 \) be defined by \( y = Tx \) with \( y_j = x_j e^{-j} \) for \( j = 1, 2, 3, \ldots \), where \( x = (x_1, x_2, ...) \) and \( y = (y_1, y_2, ...) \). Prove that \( T \) is a compact operator on \( \ell^2 \).

5. (a) Let \( f(x) = |x| \). Find its first and second derivatives using operational calculus, that is, the theory of distributions. (b) Using Green’s functions, find the solution to \( \frac{d^2 u}{dx^2} = f(x), \ u(0) = u(1) = 0 \).

Use the reproducing property of the delta distribution to formally verify that your solution satisfies the equation.
Part B.

1. Formulate and prove the Maximum Modulus Theorem.

2. Suppose \( f(z) \) is analytic in some neighborhood of the point \( z_0 \) without the point \( z_0 \) itself; let \( z_0 \) be an essential singularity of \( f(z) \).

Show that for any complex number \( C \), there is a sequence of points \( z_n \quad (n = 0, 1, 2, \ldots) \) such that

\[
 z_n \to z_0 \quad \text{and} \quad f(z_n) \to C.
\]

(In other words, in every neighborhood of an essential singularity, the function \( f(z) \) is arbitrary close to every complex number.)

3. Evaluate the integral

\[
 I = \int_{-\infty}^{+\infty} e^{iax^2} \, dx
\]

(\( a \) is a positive parameter).

4. Evaluate the integral

\[
 I = \int_{0}^{\infty} \frac{x^{\alpha} \, dx}{(x^2 + 1)}
\]

(where \( \alpha \) is a parameter, such that the integral converges).

5. Obtain the first two terms of the asymptotic expansion of

\[
 I(k) = \int_{0}^{5} \frac{e^{-kt}}{\sqrt{t^2 + 2t}} \, dx, \quad k \text{ is real and } k \to +\infty.
\]