PhD Preliminary Qualifying Examination: Applied Mathematics

August 19, 2008

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Find a singular value decomposition and pseudoinverse of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix}.$$

2. (a) Explain what is meant by a Cauchy sequence, Completeness of a space, and a Hilbert space (you can assume the definitions of norm, inner product, and linear space are known).

   (b) Prove that the set of continuous functions on the interval $[a, b]$ is complete with respect to the uniform norm

$$||f|| = \max_{x \in [a,b]} |f(x)|.$$

3. Prove that if a linear operator K can be approximated (in the operator norm) by a sequence of compact operators K_n, with ,

$$\lim_{n \to \infty} ||K_n - K|| = 0,$$

   then K is a compact operator.

4. On the interval $[0, 1]$ use eigenfunctions to find the resolvent of the integral equation

$$u(x) - \lambda \int_0^1 x^2 y^2 u(y) dy = f(x).$$

   When does a solution not exist? Write down the pseudoresolvent in this case.

5. (a) Using Green's functions, find the solution to the equation

$$u'' = f(x), \qquad u(0) = 0, \quad u(1) = 1.$$

   (b) Find the necessary and sufficient condition on $f(x)$ for a solution to

$$u'' + m^2 \pi^2 u = f(x), \qquad u(0) = u(1) = 0$$

   to exist.

Part B.

1. Find a solution $u(x,y)$ of Laplace's equation in 2 dimensions for which $u(x,0) = |x|^{1/2}$.

2. Evaluate the integral
$$I = \int_0^\infty \frac{\cos ax}{(x+1)^2} dx.$$

3. Solve the integral equation
$$\int_{-\infty}^{\infty} k(x-y)u(y)dy - u(x) = f(x)$$
with $k(x) = H(x)$, the Heaviside function.

4. Solve the system of equations
$$\frac{du_n}{dt} = \frac{1}{h^2}(u_{n+1} - 2u_n + u_{n-1}), \qquad -\infty < n < \infty$$
with $u_n(t=0) = \sin \frac{2\pi n}{k}$, with k an integer.

5. Find the leading term of the asymptotic expansion of the integral
$$I(s) = \int_{-\infty}^{\infty} e^{is(t+\frac{t^3}{3})} dt, \qquad s \text{ is real and } s \to +\infty,$$
as well as a rigorous estimate of the error.