PhD Preliminary Qualifying Examination:
Applied Mathematics
August 19, 2008

Instructions: Answer three questions from part A and three questions from part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Find a singular value decomposition and pseudoinverse of the matrix

\[ A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \end{pmatrix} . \]

2. (a) Explain what is meant by a Cauchy sequence, Completeness of a space, and a Hilbert space (you can assume the definitions of norm, inner product, and linear space are known).
(b) Prove that the set of continuous functions on the interval \([a, b]\) is complete with respect to the uniform norm

\[ \|f\| = \max_{x \in [a, b]} |f(x)|. \]

3. Prove that if a linear operator \(K\) can be approximated (in the operator norm) by a sequence of compact operators \(K_n\), with

\[ \lim_{n \to \infty} \|K_n - K\| = 0, \]

then \(K\) is a compact operator.

4. On the interval \([0, 1]\) use eigenfunctions to find the resolvent of the integral equation

\[ u(x) - \lambda \int_0^1 x^2 y^2 u(y) dy = f(x). \]

When does a solution not exist? Write down the pseudo-resolvent in this case.

5. (a) Using Green’s functions, find the solution to the equation

\[ u'' = f(x), \quad u(0) = 0, \quad u(1) = 1. \]

(b) Find the necessary and sufficient condition on \(f(x)\) for a solution to

\[ u'' + m^2 \pi^2 u = f(x), \quad u(0) = u(1) = 0 \]

to exist.
1. Find a solution $u(x, y)$ of Laplace's equation in 2 dimensions for which $u(x, 0) = |x|^{1/2}$.

2. Evaluate the integral

$$I = \int_0^\infty \frac{\cos ax}{(x + 1)^2} dx.$$ 

3. Solve the integral equation

$$\int_{-\infty}^\infty k(x - y)u(y)dy - u(x) = f(x)$$

with $k(x) = H(x)$, the Heaviside function.

4. Solve the system of equations

$$\frac{du_n}{dt} = \frac{1}{h^2} (u_{n+1} - 2u_n + u_{n-1}), \quad -\infty < n < \infty$$

with $u_n(t = 0) = \sin \frac{2\pi n}{k}$, with $k$ an integer.

5. Find the leading term of the asymptotic expansion of the integral

$$I(s) = \int_{-\infty}^{\infty} e^{is(t + \frac{t^2}{4})} dt, \quad s \text{ is real and } s \rightarrow +\infty,$$

as well as a rigorous estimate of the error.