PhD Preliminary Qualifying Examination
Applied Mathematics
Tuesday, August 16, 2016

Instructions: Answer three questions from Part A and three questions from Part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Consider the operator $T : \ell^2 \to \ell^2$ defined for $x = (\xi_j) \in \ell^2$ by
   
   $$Tx = (0, \xi_1\alpha_1, \xi_2\alpha_2, \ldots),$$

   for some sequence $(\alpha_j)$ with $\alpha_j \to 0$. Show that $T$ is compact.

2. Let $X$ be a Banach space and consider the non-linear mapping $F : X \to X$ defined by
   
   $$F(x) = y - \alpha \|x\| x,$$

   where $y \in X$ is fixed and $\alpha$ is a scalar. Show that there is a constant $C > 0$ such that for any $|\alpha| < C$, $F$ is a contraction on the open ball of radius 1 centered at $y$.

3. Let $M$ be a subset of a Hilbert space $H$. Assume $M$ is such that for any $v, w \in H$ for which the equality
   
   $$\langle v, x \rangle = \langle w, x \rangle$$

   holds for all $x \in M$, we must have $v = w$. Show that $M^\perp = \{0\}$.

4. Let $X$ be a real Banach space. Assume $f \in X^*$ has a closed nullspace $N(f)$. The goal of this problem is to show that $f$ must be a bounded linear functional.
   
   (a) Let $x_0 \in X$ be such that $f(x_0) = 1$. Explain why there is an $\epsilon > 0$ for which the ball
       
       $$B(x_0, \epsilon) = \{x \in X \mid |x - x_0| < \epsilon\}$$

       satisfies $B(x_0, \epsilon) \subset X - N(f)$.

   (b) Prove that $f(x) > 0$ for all $x \in B(x_0, \epsilon)$, where $\epsilon$ is as in part (a).

   To prove this, you may assume for contradiction that there is some $y \in B(x_0, \epsilon)$ with $f(y) < 0$.

   (c) Any $x \in B(x_0, \epsilon)$ can be written as $x = x_0 + \epsilon u$ for some $u$ with $\|u\| < 1$.

   Use the result of part (b) to show that $|f(u)| < 1/\epsilon$.

   (d) To conclude, give an upper bound for $\|f\|$.

5. Let $T : X \to X$ be a bounded linear operator on a complex Banach space $X$.

   Prove that $\sigma(T)$ lies in the complex plane disk: $\{\lambda \in \mathbb{C} \mid |\lambda| \leq \|T\|\}$.
Part B.

1. The following function \( f(x) \) \( x \) is a real variable\] can be represented by a series (Taylor or Laurent) in powers of \((x - 7)\). Find the radius of convergence of the series in each case

\[ \begin{align*}
(a) \quad f(x) &= e^{(x-7)^{10}}, \\
(b) \quad f(x) &= \left(\frac{\sin x - 3}{\sin x - 2}\right)^2, \\
(c) \quad f(x) &= \frac{\sin x - 2}{x^2 - 49}. 
\end{align*} \]

[You do not need to find the series themselves.]

2. (a) Prove: If a function is analytic then its real and imaginary parts are harmonic.

(b) Prove: If a function \( u(x, y) \) is harmonic in a domain \( D \), then \( u(x, y) \) cannot attain a strict local maximum in \( D \).

[Hint: If \( f(z) = u + iv \), then function \( \phi(z) = e^{f(z)} \) has absolute value \( |\phi(z)| = e^{u} \).]

3. Integrate

\[ \begin{align*}
(a) \quad \int_0^\infty \frac{\sin \alpha x}{x} \, dx, \\
(b) \quad \int_0^\infty \frac{x^\alpha}{1 + x} \, dx, \\
(c) \quad \int_0^\infty \sin x^2 \, dx 
\end{align*} \]

[Explain the logic (in particular, the choice of contour), but do not worry about getting the exact numbers; \( \alpha \) is a real parameter.]

4. (a) Consider a 2-dimensional map \((x, y) \rightarrow (u, v)\).

Explain: If the map is given by an analytic function \([u + iv = f(x + iy)]\) and at some point \( z_0 = x_0 + iy_0 \) the derivative \( f'(z_0) \neq 0 \), then this map preserves small shapes in the vicinity of \( z_0 \).

(b) Is a square conformally equivalent to a circle?

(c) Explain: A map by analytic function preserves Laplace’s equation.

5. Solve Laplace’s equation

\[ u_{xx} + u_{yy} = 0 \quad \text{in the domain} \quad D = \{(x, y) \in \mathbb{R}^2 : \quad (x - 1)^2 + y^2 > 1 \quad \& \quad (x - 2)^2 + y^2 < 4\} \]

subject to the boundary condition

\[ u(x, y) = a \quad \text{when} \quad (x - 1)^2 + y^2 = 1 \quad \text{and} \quad u(x, y) = b \quad \text{when} \quad (x - 2)^2 + y^2 = 4 \]

[a and b are real parameters; the two circles touch each other; express your answer in terms of the original real variables \( x, y \)].