PhD Preliminary Qualifying Examination

Applied Mathematics

August 17, 2015

Instructions: Answer three questions from Part A and three questions from Part B. Indicate clearly which questions you wish to have graded.

Part A.

1. Assume a bounded, self-adjoint operator $T : H \to H$, where H is a Hilbert space, satisfies

$$\langle Tx, x\rangle \geq \beta \|x\|^2, \tag{1}$$

for some $\beta > 0$. The goal of this problem is to show that T is one-to-one and onto.

(a) Show that the nullspace of T is trivial, i.e. $\mathcal{N}(T) = \{0\}$.

(b) Show that the range $\mathcal{R}(T)$ is closed.

(c) Show that $\mathcal{R}(T)^{\perp} = \{0\}$.

(d) Why do we have $\mathcal{R}(T) = H$?

2. Let X be a Banach space, $f : X \to \mathbb{R}$ be a bounded linear functional, $y \in X$ fixed and $\alpha \in \mathbb{R}$.

(a) Prove that there exists a constant $C > 0$ such that if $|\alpha| < C$, then the non-linear equation

$$x + \alpha f(x)x = y, \tag{2}$$

has a unique solution x in the ball $B = \{x \in X \mid \|x - y\| \leq 1\}$.

(b) Suggest an iterative procedure for approximating the unique solution x to equation (2).

3. Let (e_n) be an orthonormal sequence of a Hilbert space H. Show that $e_n \to 0$ **weakly**.

4. Let (λ_j) be a sequence real numbers with $\lambda_j \neq 1$ for all j and $\lambda_j \to 1$. Consider the operator $T : \ell^2 \to \ell^2$ defined for $(\xi_j) \in \ell^2$ by

$$T(\xi_j) = (\lambda_j \xi_j).$$

(a) Find the spectrum $\sigma(T)$, point spectrum $\sigma_p(T)$, continuum spectrum $\sigma_c(T)$, residual spectrum $\sigma_r(T)$ and resolvent set $\rho(T)$.

(b) Give a condition on the λ_j for T to be invertible.

5. Consider the distribution $u \in \mathcal{D}'(\mathbb{R})$ defined for $\phi \in \mathcal{D}(\mathbb{R})$ by

$$\langle u, \phi\rangle = \int_{-\infty}^{\infty} |x|\phi(x)dx.$$

Find its derivative ∂u.

Part B.

1. The following functions $f(x)$ [x is a real variable] can be represented by series (Taylor or Laurent) in powers of $(x-2)$. Find the radius of convergence of the series in each case

$$\begin{array}{llcl} \text{(a)} & f(x) & = & e^{x^2}\,, \\ \text{(b)} & f(x) & = & \dfrac{\sin x - 3}{\sin x - 2}\,, \\ \text{(c)} & f(x) & = & \dfrac{e^x}{(x-2)(1+e^x)}\,. \end{array}$$

2. Integrate

$$\int_{-\infty}^{\infty} \frac{\sin \alpha x}{\sinh \pi x}\, dx\,.$$

3. Solve Laplace's equation

$$\phi_{xx} + \phi_{yy} = 0 \quad \text{in the domain} \quad D = \{(x,y) \in \mathbb{R}^2 :\; x^2 + y^2 > 1,\; x^2 + (y+1)^2 < 4\}$$

subject to the boundary condition

$$\phi(x,y) = a \text{ when } x^2 + y^2 = 1 \quad \text{and} \quad \phi(x,y) = b \text{ when } x^2 + (y+1)^2 = 4$$

(a and b are real parameters; the two circles touch each other; express your answer in terms of the original real variables x ans y).

4. (a) Calculate the Fourier transform of a Gaussian $f(x) = e^{-x^2}$ $(x \in \mathbb{R})$.
   (b) Is it possible that a function $f(x)$ $(x \in \mathbb{R})$ and its Fourier transform $F(\mu)$ $(\mu \in \mathbb{R})$ *both* have finite support ?

5. Find the three-term asymtotic expansion of the integral

$$I(s) = \int_0^1 \ln(t)\, e^{ist}\, dt$$

with large real parameter $s \to +\infty$.