Show all your work, and provide reasonable justification for your answers. If you skip a step or omit some details in a proof, point out the gap and, if possible, indicate what would be required to fill it in. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Prove that $a^{561} \equiv a \pmod{561}$ for each integer $a$.

2. Determine the integers $n$ for which there exists a surjective homomorphism of symmetric groups $S_{n+1} \rightarrow S_n$.

3. Suppose $G$ is a finite group with $\text{Aut} G$ solvable. Prove that $G$ is solvable.

4. Suppose a finite group $G \neq \{e\}$ has $c$ conjugacy classes. Prove that $G$ contains an element other than the identity of order at most $c$.

5. Let $R = \mathbb{Q}[x]$, and let $M$ be the cokernel of the map

$$R^2 \xrightarrow{\begin{pmatrix} x+1 & 0 \\ 2x & x^2 \\ 1 & 1 \end{pmatrix}} R^3.$$

Write $M$ as a direct sum of cyclic $R$-modules.

6. Determine, up to conjugacy, all real $3 \times 3$ matrices $A$ satisfying $A^8 = I$ and $A^4 \neq I$.

7. Let $p$ be a prime number and $n$ a positive integer. How many elements $\alpha$ are in $\mathbb{F}_p(\alpha)$ such that $\mathbb{F}_p(\alpha) = \mathbb{F}_{p^k}$?

8. Let $k$ be a field and $k(x)$ the field of rational functions over $k$, in the indeterminate $x$. Consider the automorphisms $\sigma$ and $\tau$ of $k(x)/k$ defined by $\sigma(x) = \frac{1}{1-x}$ and $\tau(x) = \frac{1}{x}$.

(a) Prove that the subgroup $G = \langle \sigma, \tau \rangle$ of $\text{Aut}(k(x)/k)$ is isomorphic to $S_3$.

(b) Prove that the fixed subfield under the action of $G$ equals $k(t)$, where $t = \frac{(x^2 - x + 1)^3}{x^2(x-1)^2}$.

9. Let $n$ a positive integer, and let $x := x_1, \ldots, x_n$ be indeterminates over $\mathbb{Q}$. For each positive integer $i$ set

$$\alpha_i = x_1^i + \cdots + x_n^i$$

(a) Clearly $\mathbb{Q}(\alpha_1, \ldots, \alpha_n) \subseteq \mathbb{Q}(s_1, \ldots, s_n)$, the field of symmetric functions in the $x$. What is the degree of this extension?

(b) Prove the characteristic polynomial of an $n \times n$ matrix $A$ over $\mathbb{Q}$ is completely determined by the elements $\text{trace}(A), \ldots, \text{trace}(A^n)$.

10. Let $E := \mathbb{Q}(\zeta + \zeta^{-1})$ where $\zeta \in \mathbb{C}$ is a primitive $n$th root of 1, for some integer $n \geq 3$. Determine the Galois group of $E$ over $\mathbb{Q}$.