University of Utah, Department of Mathematics
January 2020, Algebra Qualifying Exam

There are ten problems on the exam. You may attempt as many problems as you wish; five correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.

1. Prove that there are at most 5 groups of order $2 \cdot 7^2$ up to isomorphism.

2. Suppose $(G,+)$ is a finite Abelian group of order $3^3 \cdot 2^3$. Suppose that $6 \cdot G = \{6x | x \in G\}$ has 6 elements. Classify G as a direct sum of cyclic groups.

3. Prove that there are no simple groups of order $2^7 \cdot 3^2$.

4. Let $R := \mathbb{Z}[x]$ and I the ideal $(3,x)$. How many elements are in $\mathrm{Ext}^i(R/I,R)$ for $i = 0,1,2$?

5. Consider $R := \mathbb{Q}[x]$ and the matrix:
$$A = \begin{bmatrix} x-1 & 0 \\ 1-x & x^2 \end{bmatrix}$$
Let M be the cokernel of the map $R^2 \xrightarrow{A\cdot} R^2$. Compute the rank of the $\mathbb{Q}$-vector space $\mathrm{Hom}_R(M, R/(x^2))$.

6. Let n be a positive integer, V a $\mathbb{C}$-vector space of rank n, and $T : V \longrightarrow V$ a $\mathbb{C}$-linear map with the property that the for each $c \in \mathbb{C}$ the eigen space $\{v \in V \mid T(v) = cv\}$ has rank at most 1.

   Prove that there exists a $w \in V$ such that the set $\{w, T(w), \ldots, T^{n-1}(w)\}$ is a basis for V.

7. With $\alpha = \sqrt{6+\sqrt{11}}$, prove that the extension $\mathbb{Q}[\alpha]/\mathbb{Q}$ is Galois and compute its Galois group.

8. Describe a primitive generator for the degree two extension of $\mathbb{Q}$ inside the extension $\mathbb{Q}(\zeta)$, where $\zeta := e^{2\pi i/7}$. Justify that the extension you identify is of degree two.

9. Describe all the prime ideals in the ring $\mathbb{Z}[x]/(x^3+1,6)$.

10. For each positive integer n let $R_n = \mathbb{Z}[2^{1/2}, \ldots, n^{1/n}]$ viewed as a subring of the field of real numbers. Prove that the ring $R := \bigcup_{n \geqslant 1} R_n$ is not Noetherian.