University of Utah, Department of Mathematics
January 2019, Algebra Qualifying Exam

There are ten problems on the exam. You may attempt as many problems as you wish; five correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.

1. Determine, up to isomorphism, the groups of order 20.

2. Is every automorphism of the alternating group A_4 an inner automorphism?

3. Let S_n denote the symmetric group on n elements. Does there exist an injective homomorphism $S_6 \longrightarrow S_5 \times S_5$?

4. Let $n \geqslant 2$ be an integer. Prove that the natural map $\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/n)$ is surjective.

5. Determine, up to conjugacy, all 3×3 matrices M over $\mathbb{Q}$ that satisfy $M^3 = 2M^2$.

6. Let M and N be $n \times n$ matrices over $\mathbb{C}$ with $MN - NM = M$. Prove that $M^k N - NM^k = kM^k$ for each $k \geqslant 1$. Prove that M is nilpotent.

7. Let $R = \mathbb{Q}[x]$ and consider the submodule M of R^2 generated by the elements $(x^2 - 1,\ x - 1)$ and $(x^2 + x,\ x)$. Write M as a direct sum of cyclic modules.

8. Identify all the prime ideals in the ring $\mathbb{Z}/14[x,y]/(2y - 1,\ x^2 + x - y^2)$.

9. Suppose that R is a commutative ring with unity, and $N \subseteq M$ are R-modules. If $\mathrm{Ext}^1(M/N, N) = 0$, prove that the inclusion $N \hookrightarrow M$ is split.

10. Let K be a field, and let L be the splitting field of an irreducible and separable polynomial $g(x) \in K[x]$. If $\mathrm{Gal}(L/K)$ is abelian, and $\alpha \in L$ is a root of $g(x)$, prove that $L = K(\alpha)$.