There are ten problems on the exam. You may attempt as many problems out of the 10 problems below as you wish; five correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.

1. Show that there is no simple group of order 112. You may use the fact that $A_7$ is simple.

2. Describe the automorphisms of the group $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$.

3. Prove that there is no element in $S_9$ of order 18.

4. Let $M$ be the $\mathbb{C}[x]$-module generated by elements $a, b, c$ modulo the three relations $a + b, x^3a + xb + xc$ and $(x^3 + 1)a + b$. Write $M$ as a direct sum of cyclic $\mathbb{C}[x]$-modules.

5. Let $R = \mathbb{Q}[x, y, z]$ and let $I = (x, y)$ and $J = (y, z)$. Compute $\text{Tor}^R_i(R/I, R/J)$ for all values of $i \geq 0$.

6. Find all prime ideals of the ring $\mathbb{Z}[x]/(15, x^3 - 2)$.

7. Prove that if a $3 \times 3$ matrix $A$ over $\mathbb{Q}$ satisfies $A^8 = I$, then $A^4 = I$. Justify claims of irreducibility of polynomials.

8. Let $F \subset L$ a extension of fields of degree 4. Prove that there are no more than 3 fields proper intermediate subfields $K$; namely, such that $F \subset K \subset L$.

9. Compute the Galois group of the polynomial $x^{10} + x^5 + 1$ over $\mathbb{Q}$.

10. Let $F$ be a field and $n$ a positive integer such that $F$ has no nontrivial field extensions of degree less than $n$. Let $L = F[\alpha]$ be an extension such that $\alpha^n$ is in $F$. Prove that each element of $L$ is a product of elements of the form $a\alpha + b$, with $a, b$ in $F$. 
