University of Utah, Department of Mathematics
January 2016, Algebra Qualifying Exam

Show all your work and provide reasonable justification. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Prove that any group of order 345 is cyclic.

2. Prove that there are no simple groups of order 90.

3. Write down the rational and Jordan canonical forms of the following matrix (viewed over $\mathbb{C}$):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -2 & 0 & 1 \\ -2 & 0 & -1 & -2 \end{bmatrix}$$

4. Prove that $n \times n$ matrices A and B have the same characteristic polynomial if and only if the trace of A^k equals the trace of B^k for each $k \geqslant 1$.

5. Let R be a Noetherian ring. Prove that any surjective ring homomorphism $\varphi\colon R \longrightarrow R$ is an isomorphism.

6. Calculate the Galois group $p(x) := x^3 + 3x + 2$, viewed as a polynomial over $\mathbb{Q}$.

7. Let K be a field and $f(x) \in K[x]$ an irreducible polynomial. Let $n \geqslant 2$ be an integer and set $g(x) := f(x^n)$. Prove that if $h(x)$ is an irreducible factor of $g(x)$, then the degree of f divides the degree of h.

8. Let p be a prime number. Prove that if $x^{p^n} - x + 1$ is irreducible over $\mathbb{F}_p$, then $n = 1$ or $n = 2 = p$.
(Hint: Note that if α is a root of the equation, then $\alpha^p - \alpha$ is in $\mathbb{F}_{p^n}$.)

9. Let $R = \mathbb{Z}[x]$, set $I = (x^3)$, and let U be the multiplicatively closed set $\{n \in \mathbb{Z} \mid n \text{ is odd}\}$. Find all the prime ideals in the ring $(U^{-1}R)/U^{-1}I$.

10. Prove that the $\mathbb{Z}$-module $\mathbb{Q}$ is not projective.