Show all your work and provide reasonable justification. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Let $\sigma$ be a 5-cycle in $S_5$. How many elements of $S_5$ commute with $\sigma$? How many elements of the subgroup $A_5$ commute with $\sigma$? How many conjugacy classes of 5 cycles are there in $A_5$?

2. Prove that, up to isomorphism, there are at most four groups of order 30.

3. Let $H$ be a proper subgroup of a finite group $G$. Prove that $\bigcup_{g \in G} gHg^{-1}$ does not equal $G$.

4. Set $G = \text{GL}_2(\mathbb{C})$. Construct a proper subgroup $H$ of $G$ such that $\bigcup_{g \in G} gHg^{-1}$ equals $G$.

5. Determine, up to conjugacy, the elements of order 4 in $\text{GL}_3(\mathbb{Q})$.

6. Let $M$ be a $3 \times 3$ matrix over $\mathbb{C}$ with trace($M^k$) = 0 for $k = 1, 2, 3$. Prove that $M$ is nilpotent.

7. Consider the polynomial ring $\mathbb{Q}[x, y]$ where $x, y$ are indeterminates over $\mathbb{Q}$. Determine a finite generating set for the ideal of polynomials $f(x, y)$ with $f(i, i) = 0$.

8. Suppose $K \subset L \subset M$ are fields with $[M : L] = 2 = [L : K]$. Prove that $M = K(\alpha)$, where $\alpha$ is a root of an irreducible polynomial in $K[x]$ of the form $x^4 + bx^2 + c$.

9. Prove that $\mathbb{Q}(\sqrt{5} + \sqrt{5})$ is Galois over $\mathbb{Q}$, and compute the Galois group.

10. Let $K$ be a field of characteristic $p > 0$, and let $t$ be an indeterminate. Consider the automorphism $\sigma \in \text{Aut}_K K(t)$ with $\sigma(t) = t + 1$. Determine the subfield of $K(t)$ that is fixed by $\sigma$. 