(1) Let $p$ be a prime. Show that an element in the symmetric group $S_n$ has order $p$ if and only if it is a product of commuting $p$-cycles. Show by an explicit example that this need not be the case if $p$ is not prime.

(2) Prove that the number of Sylow $p$-subgroups of $GL_2(\mathbb{F}_p)$ is $p + 1$.

(3) Let $G$ be a $p$-group with $|G| > p$. Show that
(a) $G$ has nontrivial center;
(b) $G$ has a normal subgroup of every order $p^m < |G|$.

(4) For the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$, find:
(a) the rational canonical form over $\mathbb{Q}$;
(b) the Jordan canonical form over $\mathbb{C}$.

(5) Prove that the ring $\mathbb{Z}[i]$ is a Euclidean domain.

(6) Let $G$ be a finite abelian group and $H$ a subgroup of $G$. Show that $G$ has a subgroup isomorphic with $G/H$.

(7) Let $m, n$ be positive integers and $d$ their greatest common divisor. Show that
$$\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}.$$

(8) For a ring $R$ define its nilradical $n(R) = \{x \in R : x^n = 0 \text{ for some } n \in \mathbb{Z}\}$.
(a) If $R$ is commutative, prove that $n(R)$ is an ideal of $R$.
(b) Is the nilradical an ideal even if $R$ is noncommutative?

(9) What is the Galois group of $x^4 - 5$ over:
(a) $\mathbb{Q}$;
(b) $\mathbb{Q}(\sqrt{5})$;
(c) $\mathbb{Q}(i)$.

(10) Show that the extension $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is Galois over $\mathbb{Q}$, and determine the Galois group.

(11) Show that the polynomial $x^2 + y^2 - 1$ is irreducible in $\mathbb{Q}[x, y]$. Is it irreducible in $\mathbb{C}[x, y]$?