PRELIMINARY EXAMINATION IN ALGEBRA

January 7, 2010

Instructions: Answer as many questions or parts of questions as you wish. A passing score consists of four complete answers or a reasonable equivalent.

1. Show that for any positive integer \( n \), every element of order 2 in the alternating group \( A_n \) is the square of an element of order 4 in the symmetric group \( S_n \).

2. Let \( G \) be a finite \( p \)-group, with \( |G| > p \). Prove that the order of \( \text{Aut}(G) \) is divisible by \( p \).

3. Let \( R \) be a ring with 1. A left \( R \)-module \( M \) is called simple if \( M \neq 0 \) and if the only submodules of \( M \) are \( M \) and 0. Show that every simple module is isomorphic to \( R/I \) for some maximal left ideal \( I \) and that \( I \) is unique if \( R \) is commutative.

4. In the category of \( \mathbb{Z} \)-modules, is the module \( \mathbb{Q}/\mathbb{Z} \) (a) projective? (b) injective? (c) flat?
   Justify your answer.

5. Let \( G \) be a group of order \( p^2q \), where \( p \) and \( q \) are distinct primes. Show that \( G \) has a normal Sylow subgroup.

6. Let \( M \) be a 5 by 5 matrix with real coefficients such that \( M^2 = 2M - I \). Show that the subspace of \( \mathbb{R}^5 \) consisting of vectors fixed by \( M \) has dimension at least 3.

7. Let \( R \) be a commutative ring with 1. Show that every \( R \)-module is free if and only if \( R \) is a field.

8. Compute the number of monic irreducible polynomials of degree 3 over the field \( \mathbb{Z}_7 \).

9. Let \( F \) be a field that contains a primitive \( n \)th root of unity. Show that if \( a \) is an element of \( F \) and the field \( E \) is obtained from \( F \) by adjoining an \( n \)th root of \( a \), then \( E \) is a Galois extension of \( F \) with cyclic Galois group.

10. State and prove Hilbert’s basis theorem.