Show all your work, and provide reasonable justification for your answers. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Determine, up to isomorphism, the groups of order 44.

2. Prove that there is no simple group of order 192.

3. Let $R = \mathbb{Q}[x, y]$ and let $I = \langle x, y \rangle$. Compute the vector space rank of $\text{Ext}^1_R(I, R)$ over $\mathbb{Q}$.

4. Let $M$ be a $3 \times 3$ complex matrix with $M^6 = M^4$ and $M^4 + M^2 = 2M^3$. Determine possible Jordan forms of $M$.

5. (i) List the prime ideals of the ring $R = \mathbb{Z}[x, y]/\langle 3 + x, y(y - 1) + x^2, x \rangle$.

(ii) Give an example of an integral domain with exactly 2 prime ideals.

6. Suppose that $R = \mathbb{Z}[i]$ is the Gaussian integers (here $i$ is a square root of $-1$). Let $M$ and $N$ be finitely generated $R$-modules such that $M \oplus R^{\oplus 2} \oplus R/\langle 2 \rangle \cong N \oplus R \oplus R/\langle 1 + i \rangle \oplus R \oplus R/\langle 1 - i \rangle$.

Is it true that $M \cong N$?

7. Prove that $\mathbb{F} = \mathbb{Z}[t]/\langle 3, t^3 - t^2 + 1 \rangle$ is a field. Find the number of solutions of $x^{13} + 1 = 0$ in $\mathbb{F}$, and also the number of solutions of $x^{13} - 1 = 0$ in $\mathbb{F}$.

8. Compute the Galois group of $x^4 - 2$ over $\mathbb{Q}$.

9. Let $E$ be the splitting field of $f(x) = x^{14} + 1$ over $\mathbb{F}_2$, and let $K$ be the splitting field $g(x) = x^{21} + 1$ over $\mathbb{F}_2$.

Prove that $K$ contains an isomorphic copy of $E$, and compute the extension degree of $K$ over $E$.

10. Let $f(x)$ be a degree 5 polynomial in $\mathbb{Q}[x]$ that is not solvable by radicals. Let $L$ be its splitting field over $\mathbb{Q}$.

(i) Prove that there is at most one field $K$ with $\mathbb{Q} \subset K \subset L$ and $[K : \mathbb{Q}] = 2$.

(ii) If $\alpha, \beta$ are irrational elements in $L$ such that $\alpha^2$ and $\beta^2$ are rational, prove that $\alpha\beta$ is rational.